TIMING PROCESSES

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ABSTRACT. We argue that monoidal lax categories are a good setting for timing processes and we conjecture a construction of the free monoidal lax category with a monoidal lax functor to the natural numbers.

1. Monoidal Lax Categories

1.1. DEFINITION. *A* monoidal lax category *is a locally posetal category* $(\mathbb{C}, \leq)$ *endowed with two lax functors,* $(\otimes)\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *and* $I\colon \mathbb{1} \to \mathbb{C}$*, that for now, we assume to be strictly associative and unital. The laxity of the tensor functor implies the following axioms, the laxity of the unit functor is redundant.*

$$(f \otimes g) \mathbin{⨾} (f' \otimes g') \geq (f \mathbin{⨾} f') \otimes (g \mathbin{⨾} g'),$$
$$\mathrm{id}_{A \otimes B} \geq \mathrm{id}_A \otimes \mathrm{id}_B.$$

1.2. EXAMPLE. The natural numbers, with the sum and the maximum, form a monoidal lax category with a single object $(\mathbb{N}, +, \max)$. Composition is given by the sum and the tensor is given by the maximum, which is only a lax tensor,

$$\max\{f; g\} + \max\{f'; g'\} \geq \max\{f \mathbin{⨾} f'; g \mathbin{⨾} g'\},$$
$$0 \geq \max\{0; 0\}.$$

1.3. DEFINITION. *A* timing doctrine *for a monoidal lax cateogry is a monoidal lax functor into* $(\mathbb{N}, +, \max)$*. This means it must be an assigment* $T\colon \mathbb{C} \to \mathbb{N}$ *satisfying the following axioms.*

$$T(f) + T(g) \geq T(f \mathbin{⨾} g), \qquad 0 \geq T(\mathrm{id}),$$
$$T(f \otimes g) \geq \max\{T(f), T(g)\}, \qquad T(\mathrm{id}_I) = 0.$$

Timing doctrines form a slice category. There is a forgetful functor from timing doctrines into weighted polygraphs that has a left adjoint constructing the free timing doctrine on some weighted generators.

1.4. REMARK. There exists a functor between the free timing doctrine over a weighted polygraph and the free monoidal category over that same polygraph. This functor, $U\colon \mathsf{Time}(\mathcal{H}) \to \mathsf{Mon}(\mathcal{H})$, returns the morphism without caring about the time; it has a right adjoint given by the minimum decomposition of the morphism in the timing doctrine – the most efficient implementation of that morphism.

1.5. Theorem. [Sketch] *The free timing doctrine over a weighted polygraph is given by the morphisms with arbitrary vertical boundary of the following free double category: the free double category with horizontal objects the same as the monoidal category; vertical objects the monoid of the natural numbers; a single cell,* $f(t \downarrow t) \colon A \to B$*, for each generator of weight* $t \in \mathbb{N}$*; and a interchange cell between wires and time, that allows to freely move a process preserving its time.*

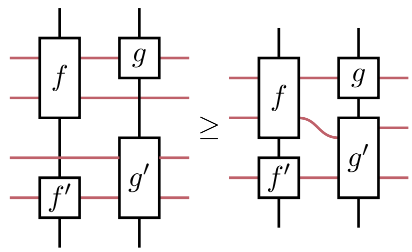

Figure 1: A morphism in the free timing doctrine.

This technical report continues the work of a previous note where this double category was detailed [1].

References

[1] Elena Di Lavore and Mario Román. Timing Processes the Naive Way. Internal technical report, Tallinn University of Technology, https://www.ioc.ee/~mroman/data/notes/timing-processes.pdf., 2020.