Be Careful with Duoidal Coherence. Monoidal categories possess a coherence theorem that determines that any two parallel morphisms constructed out of structure isomorphisms commute. In contrast, duoidal categories do not satisfy that same statement. This causes some confusion around coherence for duoidal categories. I bring an example of how this confusion may arise, hoping that it will help the interested reader and that it may further justify the importance of expository category theory.

We could be tempted to provide an alternative definition of duoidal categories that avoids asking for a bunch of commutative diagrams by simply asking that any formal such diagram commutes. This proposal I take from Spivak and Shapiro [SS22], who comment that

> *alternatively, duoidal categories can be defined by the two monoidal structures along with the generating structure maps [...] (4) natural in a, b, c, d which satisfy equations guaranteeing that any two structure maps built from those in (4) between the same two expressions in $y_\otimes, y_\triangleleft, \otimes, \triangleleft$ are equal.*

One of the first, most complete and comprehensive accounts of duoidal categories is the monograph by Aguiar and Mahajan [AM10]. It includes a passing comment that could suggest that this version can be proven correct. It says that

> *"[...] if two morphisms $A \to B$ are constructed out of the structure maps in $\mathbb{C}$ (including the structure constraints of the monoidal categories $(\mathbb{C}, \diamond, I)$ and $(\mathbb{C}, \star, J)$), then they coincide."*

However, intepreted literally, this turns out to not be true. Two parallel morphisms constructed out of the structure maps of a duoidal category do not need to coincide.

Proposition 2.6. *There may exist two different maps of type $I \triangleleft I \to I$ constructed out of the structure maps of a duoidal category.*

Proof. We can consider two maps of type $I \triangleleft I \to I$, depending on which of the two parallel units we decide to convert to a sequential unit using the laxators. Explicitly, we are saying that $(I \triangleleft \varphi_0) ⨟ \rho_\triangleleft$ and $(\varphi_0 \triangleleft I) ⨟ \lambda_\triangleleft$ do not coincide. We construct an example of this phenomenon.

Consider the duoidal category of endoprofunctors over a monoidal category. This is one of the first examples of duoidal category described by Street [Str12]; it is also described by Garner and López Franco [GF16], even when the axioms are not explicitly checked in print.

[TODO: Diagram]

In this category of endoprofunctors over $\mathbb{C}$, parallel tensor is the profunctor $I(X; Y) = \mathbb{C}(X; I) \times \mathbb{C}(I; Y)$, and sequencing two of them gives

$$(I \triangleleft I)(X; Y) = \hom(X; I) \times \hom(I; I) \times \hom(I; Y).$$

In this case, the two maps send the triple (f, a, g) to $(f ⨟ a, g)$ and $(f, a ⨟ g)$, respectively. However, these two pairs do not need to be equal if $a \in \hom(I; I)$ is a non-identity morphism. □

Example 2.7 (Graded spaces)**.** We look for a more classical source of examples in the theory of *graded spaces*. Let $(\mathbb{V}, \otimes, I)$ be a monoidal category with coproducts that are preserved by the tensor; let $(G, +, 0)$ be a commutative monoid. We say that the functor category $[G, \mathbb{V}]$ is the category of *G-graded $\mathbb{V}$-spaces*. This category has a rich structure; we highlight two of its tensor products: the *pointwise* or *Hadamard* tensor product

$$(V \otimes W)_n = V_n \otimes W_n, \text{ for each } n \in G, \text{ with unit } \mathbb{I}_n = I;$$

and the *convolution* or *Cauchy* tensor product

$$(V \bullet W)_n = \sum\nolimits_{k+m=n} V_k \otimes W_m, \text{ with unit } \mathbf{1}_n = \mathbf{0} \text{ except for } \mathbf{1}_0 = I.$$

These two tensors interact in a duoidal category with a laxator as follows; see for instance the work of López Franco and Vasilakopoulou [FV20].

$$\sum_{k+m=n} V_k \otimes W_k \otimes U_m \otimes Z_m \to \left(\sum_{k_1+m_1=n_1} V_{k_1} \otimes U_{m_1} \right) \otimes \left(\sum_{k_2+m_2=n_2} W_{k_2} \otimes Z_{m_2} \right).$$

Proposition 2.8. *Dually, there exist two different maps of type $J \to J \otimes J$ constructed out of the structure maps of a duoidal category.*

Proof. This follows from the previous Proposition 2.6, by considering the opposite duoidal category. However, let us comment a second example [AM]. Consider the duoidal category of graded spaces over a monoid G. The two maps, $\mathbb{I} \to \mathbb{I} \otimes 1 \to \mathbb{I} \otimes \mathbb{I}$ and $\mathbb{I} \to 1 \otimes \mathbb{I} \to \mathbb{I} \otimes \mathbb{I}$, correspond to inclusions of the vector space graded by $g \in G$ into the summand indexed by $(g, 0)$ or $(0, g)$, respectively; these are different in general. □

In fact, the stronger statement of cohernece does not seem to be used explicitly in any of these two texts, and the definition of duoidal categories as completely coherent strucutres is not usually found in the literature. Most authors, like Aguiar and Mahajan [AM10], and Garner and López Franco [GF16], revert to the definition of duoidal category as a 2-monoid in the monoidal bicategory of monoidal categories.

Aguiar and Mahajan [AM10] do actually point out that the expected coherence theorem should follow from the coherence theorem for lax monoidal functors. The confusion can arise if one does not realize that this coherence theorem does not actually prove that any two parallel maps coincide: in particular, coherence for lax monoidal functors does not prove that the two maps $F(I) \otimes F(I) \to F(I)$ coincide. In this case, however, the problem is better known – it is mentioned by Malkiewich and Ponto [MP21], who cite a short mention in the original proof by Lewis [Lew06] and Kelly and Laplaza [KL80].

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