COALGEBRAIC STOCHASTIC CONTINUOUS SYSTEMS

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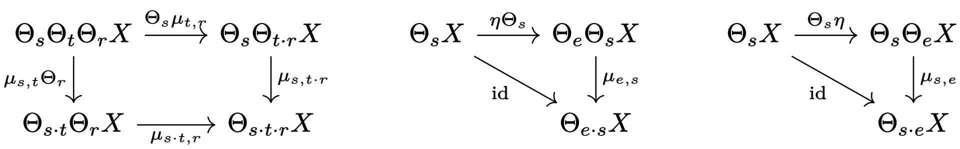
ABSTRACT. We argue for the time-graded coalgebras of the Giry monad to be suitable coalgebraic stochastic continuous dynamical systems.

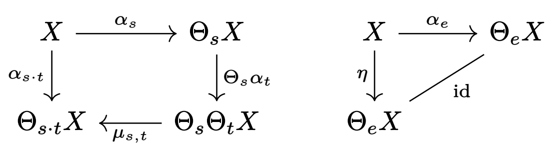
Definition 1 (Graded monad)**.** A *graded monad* Θ in a category $\mathbb{C}$, graded on a monoid $(M, \cdot, e)$, is a family of endofunctors $\Theta_s \colon \mathbb{C} \to \mathbb{C}$ together with a family of natural transformations,

$$\mu_X^{s,t} \colon \Theta_s(\Theta_t(X)) \to \Theta_{s \cdot t}(X);$$

and a natural transformation $\eta_X \colon X \to \Theta_e(X)$; making the following diagrams commute.



Definition 2 (Graded coalgebra for a graded monad)**.** A *graded coalgebra* for a graded monad Θ is a carrier object, X, together with a family of morphisms $\alpha s \colon X \to \Theta_s(X)$ indexed over the monoid $(M, \cdot, e)$, and making the following diagrams commute.



Definition 3 (Homomorphism of graded coalgebras)**.** A homomorphism between two graded coalgebras, $\gamma_1^s \colon X \to \Theta_s(X)$ and $\gamma_2^s \colon Y \to \Theta_s(Y)$, is a family of morphisms $f \colon X \to Y$ satisfying $\gamma_1^s \fatsemi \Theta_s(f) = f \fatsemi \gamma_2^s$.

Example 4 (Brownian motion). Consider the Giry monad [Gir82] as trivially graded over the real numbers, $\Theta_s = D$. The family of morphisms $\beta_s \colon X \to D(X)$ defined by $\beta_s(x) = \mathrm{Normal}(x; s)$ forms a graded coalgebra for the Giry monad. Proving this amounts to check that, if $y \sim \mathrm{Normal}(x; s)$ and $z \sim \mathrm{Normal}(y; t)$, then $z \sim \mathrm{Normal}(x; s + t)$; and that $y \sim \mathrm{Normal}(x; 0)$ means $y = x$.

REFERENCES

[Gir82] Michèle Giry. A categorical approach to probability theory. In *Categorical aspects of topology and analysis*, pages 68–85. Springer, 1982.

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