3. Distributive Laws of Promonads

3.1. **Promonads.** Promonads are promonoids in the double category of categories, but they can be characterized simply as identity-on-object functors. In some sense, they endow a category with new morphisms, new hom-sets given by a profunctor: the unit of the promonad transforms old morphisms into new morphisms, and the multiplication represents composition.

Definition 3.1. A *promonad* $(\mathbb{C}, \star, ^\circ)$ over a category $\mathbb{V}$ is a profunctor $\mathbb{C} \colon \mathbb{V} \nrightarrow \mathbb{V}$ together with natural transformations representing inclusion $(^\circ)_{X,Y} \colon \mathbb{V}(X,Y) \to \mathbb{C}(X,Y)$ and multiplication $(\star)_{X,Y,Z} \colon \mathbb{C}(X,Y) \times \mathbb{C}(Y,Z) \to \mathbb{C}(X,Z)$, and such that

i. the right action is premultiplication, $f^\circ \star p = f > p$;
ii. the left action is posmultiplication, $p \star f^\circ = p < f$;
iii. multiplication is dinatural, $p \star (f > q) = (p < f) \star q$;
iv. and multiplication is associative, $(p_1 \star p_2) \star p_3 = p_1 \star (p_2 \star p_3)$.

Equivalently, promonads are identity on objects functors, and their main examples are the Kleisli categories of monads and comonads.

Proposition 3.2. *Every monad* (T, μ, η) *induces a promonad* $(\mathbb{T}, \star, ^\circ)$.

Proof. Let us define $\mathbb{T}(A,B) = \hom(A, TB)$. This is a profunctor with the actions $u > f = u \mathbin{⨾} f$ and $f < v = f \mathbin{⨾} Tv$. We define $f \star g = f \mathbin{⨾} Tg \mathbin{⨾} \mu$ and $u^\circ = u \mathbin{⨾} \eta$. □

Proposition 3.3. *Every comonad* (D, δ, ε) *induces a promonad* $(\mathbb{D}, \star, ^\circ)$.

Proof. Let us define $\mathbb{D}(X,Y) = \hom(DX, Y)$. This is a profunctor with the actions $u > f = Du \mathbin{⨾} f$ and $f < v = f \mathbin{⨾} v$. We define $f \star g = \delta \mathbin{⨾} Df \mathbin{⨾} g$ and $u^\circ = \varepsilon \mathbin{⨾} u$. □

3.2. **Distributive laws.** We can define distributive laws among monads, among comonads, but also between comonads and monads. All of these need to be axiomatized separately, but all of them give rise to new Kleisli categories.

We introduce the notion of a distributive law of promonads. Promonads are a way of unify the axiomatization of Kleisli categories without caring whether they come from a monad or a comonad. The different notions of distributive law can be shown to be particular cases of this more general notion.

Definition 3.4. A *distributive law* of a comonad (D, δ, ε) over a monad (T, μ, η) is a natural transformation $\lambda_X \colon DTX \to TDX$ satisfying

(1) $D\eta \mathbin{⨾} \lambda = \eta$, and $\lambda \mathbin{⨾} T\lambda \mathbin{⨾} \mu = D\mu \mathbin{⨾} \lambda$,
(2) $\lambda \mathbin{⨾} T\varepsilon = \varepsilon$, and $\delta \mathbin{⨾} D\lambda \mathbin{⨾} \lambda = \lambda \mathbin{⨾} T\delta$.

Definition 3.5. A *distributive law* between two promonads $(\mathbb{C}, \star, ^\circ)$ and $(\mathbb{D}, \star, ^\circ)$ over the same category $\mathbb{V}$ is a family of functions

$$d_{A,B,C} \colon \mathbb{D}(A,B) \times \mathbb{C}(B,C) \to \int^{B'} \mathbb{C}(A,B') \times \mathbb{D}(B',C),$$

natural in A and C, dinatural in B, quotiented by dinaturality of B', and separately preserving identities and composition.

In other words, it must satisfy the following axioms.

(1) left naturality, $d(u > f \mid g) = \text{let } d(f \mid g) \to (g' \mid f') \text{ in } (u > g' \mid f')$,
(2) right naturality, $d(f \mid g < u) = \text{let } (g' \mid f') \to d(f \mid g) \text{ in } (g' \mid f' < u)$,
(3) left unitality, $d(u^\circ \mid g < v) = (u > g \mid v^\circ)$,
(4) right unitality, $d(u < f \mid v^\circ) = (u^\circ \mid f < v)$,

(5) left compositionality,

$$\begin{aligned} d(q_1 \star q_2 \mid p) = &\text{let } d(q_2 \mid p) \to (p' \mid q_2') \text{ in} \\ &\text{let } d(q_1 \mid p') \to (p'' \mid q_1') \text{ in} \\ &(p'' \mid q_1' \star q_2') \end{aligned}$$

(6) and right compositionality,

$$\begin{aligned} d(q_1 \star q_2 \mid p) = &\text{let } (p' \mid q_2') \leftarrow d(q_2 \mid p) \text{ in} \\ &\text{let } (p'' \mid q_1') \leftarrow d(q_1 \mid p') \text{ in} \\ &(p'' \mid q_1' \star q_2'). \end{aligned}$$

Proposition 3.6. *A distributive law, $\lambda_X \colon DTX \to TDX$, of the comonad (D, δ, ε) over the monad (T, μ, η) induces a distributive law between their corresponding Kleisli promonads $\mathbb{D}$ and $\mathbb{T}$.*

Proof. We define the transformation $d \colon \mathbb{T} * \mathbb{D} \to \mathbb{D} * \mathbb{T}$ to be

$$d(f \mid g) = (Df \fatsemi \lambda \mid Tg) = (Df \mid \lambda \fatsemi Tg).$$

We will now check that this distributive law satisfies the axioms.

(1) left naturality,

$$d(u > f \mid g) = d(u \fatsemi f \mid g) = (Du \fatsemi Df \mid \lambda \fatsemi Tg) = (u > Df \mid \lambda \fatsemi Tg),$$

(2) right naturality,

$$d(f \mid g < u) = d(f \mid g \fatsemi u) = (Df \fatsemi \lambda \mid Tg \fatsemi Tu) = (Df \fatsemi \lambda \mid Tg < u),$$

(3) left unitality,

$$\begin{aligned} d(u^\circ \mid g < v) &= d(u \fatsemi \eta \mid g \fatsemi v) = (Du \fatsemi D\eta \fatsemi \lambda \mid Tg \fatsemi Tv) \\ &= (Du \fatsemi \eta \mid Tg \fatsemi Tv) = (Du \fatsemi g \mid v \fatsemi \eta) = (u > g \mid v^\circ), \end{aligned}$$

(4) right unitality,

$$\begin{aligned} d(u > f \mid v^\circ) &= d(u \fatsemi f \mid \varepsilon \fatsemi v) = d(Du \fatsemi Df \mid \lambda \fatsemi T\varepsilon \fatsemi Tv) \\ &= (Du \fatsemi Df \mid \varepsilon \fatsemi Tv) = (\varepsilon \fatsemi u \mid f \fatsemi Tv) = (u^\circ \mid f < v), \end{aligned}$$

(5) left compositionality,

$$\begin{aligned} d(f_1 \star f_2 \mid g) &= d(f_1 \fatsemi Tf_2 \fatsemi \mu \mid g) = (Df_1 \fatsemi DTf_2 \fatsemi D\mu \fatsemi \lambda \mid Tg) \\ &= (Df_1 \fatsemi DTf_2 \fatsemi \lambda \fatsemi T\lambda \fatsemi \mu \mid Tg) \\ &= (Df_1 \fatsemi \lambda \fatsemi TDf_2 \fatsemi T\lambda \fatsemi \mu \mid Tg) \\ &= (Df_1 \fatsemi \lambda \mid TDf_2 \fatsemi T\lambda \fatsemi \mu \fatsemi Tg) \\ &= (Df_1 \fatsemi \lambda \mid TDf_2 \fatsemi T\lambda \fatsemi TTg \fatsemi \mu), \end{aligned}$$

(6) right compositionality,

$$\begin{aligned} d(f \mid g_1 \star g_2) &= d(f \mid \delta \fatsemi Dg_1 \fatsemi g_2) = (Df \mid \lambda \fatsemi T\delta \fatsemi TDg_1 \fatsemi Tg_2) \\ &= (Df \mid \delta \fatsemi D\lambda \fatsemi \lambda \fatsemi TDg_1 \fatsemi Tg_2) \\ &= (Df \mid \delta \fatsemi D\lambda \fatsemi DTg_1 \fatsemi \lambda \fatsemi Tg_2) \\ &= (Df \fatsemi \delta \fatsemi D\lambda \fatsemi DTg_1 \mid \lambda \fatsemi Tg_2) \\ &= (\delta \fatsemi DDf \fatsemi D\lambda \fatsemi DTg_1 \mid \lambda \fatsemi Tg_2). \end{aligned}$$

This proves that our definition determines a distributive law. □