SOLVING PUZZLES IN DECISION THEORY

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Abstract. We solve a sample of puzzles from probabilistic decision theory using the operational semantics of the subdistribution monad to interpret do-notation statements as mathematical objects.

Contents

1. Introduction

Decision theory problems are famously controversial: the Monty Hall controversy [Sel75, vS], Newcomb's paradox [Noz69], or the sleeping beauty paradox [Elg20] all have been much debated both in philosophy and mathematics.

At the same time, even formalizing the statement of simple problems in statistical inference causes a reasonable amount of confusion. Recently, Jacobs conducted a small survey among a hundred academic colleagues in AI and medical statistics: the answers were inconsistent even inside the same field; the approach and notation were not systematic [Jac24]. It is fair to say that a procedure for solving probabilistic decision theory problems is not common knolwedge. We could blame the implicit assumptions hidden in problem statements: a good formal language that makes these assumptions explicit could help us settle down these controversies.

We propose a simple syntax and semantics for inference and decision problems. The semantics is simple enough to be followed easily with pen and paper, and we provide explicit examples.

1.1. **Example: the Monty-Hall problem.** The Monty Hall problem first appeared in a letter by Steve Selvin to the editor of the *American Statistician* in 1975 [Sel75]. However, it was vos Savant's discussion in *Parade* magazine, prompted by a reader, that brought controversy and fame to the problem [vS].

Key words and phrases. Category theory, categorical semantics.

> *Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say #1, and the host, who knows what's behind the doors, opens another door, say #3, which has a goat. He says to you, "Do you want to pick door #2?" Is it to your advantage to switch your choice of doors?* — Craig F. Whitaker, at "Ask Marylin" [vS].

Let us formalize and solve the Monty-Hall problem. Without loss of generality, we simplify the problem assuming that we choose the Middle door and that the host opened the Left door. The problem states that: (1) the car is uniformly distributed along the three doors (Left, Middle, and Right); (2) the host, knowing where the car is, opens a door where the car is not; (3) we observe that the host opened the left door; (4) we want to know where the car is. In Figure 1, we write and solve the problem.

The important bit here is that, once we agree on the four statements describing the problem, the rest is calculational: we should all agree on its resolution. In particular, under this statement, we obtain that the posterior, $\frac{1}{3}M + \frac{2}{3}R$, gives double probability to the car being on the Right door: if we want to win the car, we should switch doors.

$$
\begin{array}{lll}
(1) & \mathit{car} \leftarrow \textsc{uniform}\{L, M, R\} & \frac{1}{3}L + \frac{1}{3}M + \frac{1}{3}R \\
(2) & \mathit{open} \leftarrow \mathrm{host}(\mathit{car}) & \cancel{\frac{1}{3}LR} + \frac{1}{3}M(\frac{1}{2}L + \cancel{\frac{1}{2}R}) + \frac{1}{3}RL \\
(3) & \textsc{observe}(\mathit{open} = L) & \frac{1}{6}ML + \frac{1}{3}RL \\
(4) & \textsc{return}(\mathit{car}) & \frac{1}{6}M + \frac{1}{3}R \\
 & & \\
 & \text{Validity:} & \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \\
 & \text{Posterior:} & \frac{1}{3}M + \frac{2}{3}R
\end{array}
$$

FIGURE 1. Solution of the Monty Hall problem.

1.2. **Synopsis.** The rest of this paper is divided in two sections. The first explains the notation we employ in Figure 1; the second solves multiple problems of decision theory: Newcomb's paradox, the "Sleeping beauty problem", the "Sailor's child" problem, or the "Death in Damascus" problem.

1.3. **Related work.** We employ the do-notation syntax of discrete partial Markov categories [DLR23]. We present this structure without explicitly mentioning its categorical inspiration, restricting ourselves to the semantics of finitary subdistributions: Gibbons and Hinze had already mentioned that Haskell's do-notation [Hug00, Pat01, HJ06] was amenable to decision problems such as Monty Hall [GH11].

Probabilistic programming has a long tradition of categorical and monadic semantics [Has97, SV13, SWY+16, HKSY17, EPT17, DK19, VKS19]; it usually employs more complex languages, not suitable for pen-and-paper computation. Our work tries to constitute a minimal setup that, while less expressive, may be easier to employ when discussing decision theory.

This paper intends to be a practical counterpoint to "Evidential Decision Theory via Partial Markov Categories" [DLR23]. We purposely avoid explicit references to category theory in this work.

2. Do-notation and its semantics

Do-notation for copy-discard categories is a variant of the do-notation employed for Haskell's arrows [HJ06, Hug00, Jef97]. It consists of a series of statements declaring the input and output variables of every function, ended by a return statement.

Definition 2.1. A *do-notation description*, over a set of context variables Γ, is inductively defined to be either

(1) a *return* statement, representing that the description has ended and that the value of some variables is consulted: the return statement consists of some of the variables of Γ, appearing in any order; e.g., $\mathrm{return}(x_1, ..., x_n)$ for $x_1, ..., x_n \in \Gamma$;
(2) a *function* statement, representing the application of a substochastic channel: the function statement consists of a list of input variables, e.g. $x_1, ..., x_n \in \Gamma$; a substochastic channel, e.g. f; and a list of fresh output variables, e.g. $y_1, ..., y_n$, together with a do-notation description over the context extended with these fresh variables, $\Gamma, y_1, \ldots, y_m$.

$$\frac{}{\Gamma \vdash \mathrm{return}(x_1, ..., x_n)} \qquad \frac{\Gamma, y_1, \ldots, y_m \vdash \mathrm{prog}\colon \Delta}{\Gamma \vdash y_1, \ldots, y_m \leftarrow \mathrm{f}(x_1, ..., x_n) \,;\, \mathrm{prog}\colon \Delta}$$

FIGURE 2. Rules for do-notation.

In summary, a do-notation description is generated by the type theory of Figure 2; there, we take the x-indexed variables to appear in the context, $x_1, ..., x_n \in \Gamma$.

Remark 2.2. In particular, in our setting we will always consider OBSERVE to be a generator with a single input of boolean type and no outputs.

$$\text{OBSERVE}(p) = \text{CASE } p \text{ OF True} \mapsto [(*, 1)]; \text{False} \mapsto [];$$

2.1. The Operational Semantics of Subdistributions.

To put it concisely, we use as operational semantics the monadic semantics of do-notation over the subdistribution monad, $\mathbf{D}_{\leq 1}\colon \mathbf{Set} \to \mathbf{Set}$.

We interpret subdistributions as formal sums whose coefficients add up to less or equal than 1. Every term of the formal sum, $\lambda X_1 \ldots X_n$, represents a probability of λ of getting the values $X_1, \ldots, X_n$ for the variables forming the context $\Gamma = x_1, \ldots, x_n$. The formal sum represents the independent addition of these subdistributions; a generic subdistribution is of the form

$$\lambda_1 X_1^1 \ldots X_n^1 + \cdots + \lambda_k X_1^k \ldots X_n^k, \qquad \text{for } \lambda_1 + \cdots + \lambda_k \leq 1.$$

Each do-notation statement $\Gamma \vdash \Delta$ translates to a function $\Gamma \to \mathbf{D}_{\leq 1}(\Delta)$. Explicitly,

(1) a return statement, $\mathrm{return}(x_1, ..., x_n)$, corresponds to a projection of the involved variables; it will map any context $\Gamma = \Gamma_1 X_1 \Gamma_2 \ldots \Gamma_n X_n \Gamma_{n+1}$ to the context $X_1 \ldots X_n$;
(2) a function statement, $y_1, \ldots, y_m \leftarrow \mathrm{f}(x_1, ..., x_n)$, corresponds to the application of the substochastic channel to a copy of the involved variables; that is, it will map any context $\Gamma = \Gamma_1 X_1 \Gamma_2 \ldots \Gamma_n X_n \Gamma_{n+1}$ to $\Gamma X_1 \ldots X_n$,

then, if the output of the function is $f(X_1, \dots, X_n) = \lambda_1 Y_1^1 \dots Y_n^1 + \cdots + \lambda_k Y_1^k \dots Y_n^k$, return a new sum of contexts with these new variables,

$$\lambda_1 \Gamma Y_1^1 \dots Y_n^1 + \cdots + \lambda_k \Gamma Y_1^k \dots Y_n^k.$$

After each statement, we can distribute the multiplications over the sums and regroup monomials. Applications of an OBSERVE statement result in a multiplication by zero, and we simply cancel that term.

3. Problem – Three Covid tests

> *Consider a disease, like Covid, say with a prevalence of 5%. This means that the chance that an arbitrary person in the population has the disease is $\frac{1}{20} = 0.05$. This is the prior disease probability. There is a test for the disease that is not perfect, as usual.*
>
> *The sensitivity of the test is 90%; this means that if a person has the disease, the probability that the test is positive (for this person) is $\frac{9}{10} = 0.9$. The specificity is 60%; this means that if a person does not have the disease, then the probability that the test is negative is $\frac{3}{5} = 0.6$. In this situation the predicted positive test probability is $\frac{17}{40} = 0.425$.* — Bart Jacobs, "Getting Wiser from Multiple Data" [Jac24].

Individuals may be ill (I) or healthy (H). We have a prior distribution, $\mathrm{prior} = \frac{1}{20}I + \frac{19}{20}H$. A test may be positive (P) or negative (N). Testing is a channel defined by $\mathrm{test}(I) = \frac{9}{10}P + \frac{1}{10}N$ and $\mathrm{test}(H) = \frac{2}{5}P + \frac{3}{5}N$.

Let us state and solve the "three Covid tests" problem. The problem states that: (1) we pick an individual from the population; (2,3) we take a first test and observe it positive; (4,5) we take a second test and observe it positive; and (6,7) we take a final third test and observe it negative. We want to know the probability of the patient being ill (8). The solution is computed in Figure 3.

$$\begin{array}{lll}
(1) & \mathit{individual} \leftarrow \mathrm{prior} & \frac{1}{20}I + \frac{19}{20}H \\
(2) & \mathit{result}_1 \leftarrow \mathrm{test}(\mathit{individual}) & \frac{1}{20}I(\frac{9}{10}P + \cancel{\frac{1}{10}N}) + \frac{19}{20}H(\frac{2}{5}P + \cancel{\frac{3}{5}N}) \\
(3) & \mathrm{OBSERVE}(\mathit{result}_1 = P) & \frac{9}{200}IP + \frac{19}{50}HP \\
(4) & \mathit{result}_2 \leftarrow \mathrm{test}(\mathit{individual}) & \frac{9}{200}IP(\frac{9}{10}P + \cancel{\frac{1}{10}N}) + \frac{19}{50}HP(\frac{2}{5}P + \cancel{\frac{3}{5}N}) \\
(5) & \mathrm{OBSERVE}(\mathit{result}_2 = P) & \frac{81}{2000}IPP + \frac{19}{125}HPP \\
(6) & \mathit{result}_3 \leftarrow \mathrm{test}(\mathit{individual}) & \frac{81}{2000}IPP(\cancel{\frac{9}{10}P} + \frac{1}{10}N) + \frac{19}{125}HPP(\cancel{\frac{2}{5}P} + \frac{3}{5}N) \\
(7) & \mathrm{OBSERVE}(\mathit{result}_3 = N) & \frac{81}{20000}IPPN + \frac{37}{625}HPPN \\
(8) & \mathrm{RETURN}(\mathit{individual}) & \frac{81}{20000}I + \frac{37}{625}H \\
 & & \\
 & \text{Validity:} & \frac{81}{20000} + \frac{37}{625} = \frac{381}{4000} \\
 & \text{Posterior:} & \frac{27}{625}I + \frac{608}{625}H
\end{array}$$

FIGURE 3. Calculations for the "three Covid tests" problem.

3.1. Variant: Unordered tests. Assuming that we do not know the order in which the tests have been taken: we only know that there are two positive tests and a single negative test. This, of course, does not alter the posterior distribution — the patient being ill is independent on the order the tests were performed — but it does alter the validity of the procedure: it is much more likely to observe a

$$
\begin{array}{lll}
(1) & \textit{individual} \leftarrow \text{prior} & \frac{1}{20}I + \frac{19}{20}H \\
(2) & \textit{result}_1 \leftarrow \text{test}(\textit{individual}) & (\frac{9}{200}IP + \frac{19}{200}IN) + (\frac{19}{50}HP + \frac{27}{250}HN) \\
(3) & \textit{result}_2 \leftarrow \text{test}(\textit{individual}) & (\frac{81}{2000}IPP + \frac{9}{2000}IPN) + (\frac{171}{2000}INP \\
 & & \quad + \frac{9}{2000}INN) + (\frac{19}{125}HPP + \frac{57}{250}HPN) \\
 & & \quad +(\frac{57}{250}HNP + \frac{171}{500}HNN) \\
(4) & \textit{result}_3 \leftarrow \text{test}(\textit{individual}) & (\cancel{\frac{729}{20000}IPPP} + \frac{81}{20000}IPPN) \\
 & & \quad +(\frac{81}{20000}IPNP + \cancel{\frac{9}{20000}IPNN}) \\
 & & \quad +(\frac{81}{20000}INPP + \cancel{\frac{9}{20000}INPN}) \\
 & & \quad +(\cancel{\frac{9}{20000}INNP} + \cancel{\frac{9}{20000}INNN}) \\
 & & \quad +(\cancel{\frac{38}{625}HPPP} + \frac{57}{625}HPPN) \\
 & & \quad +(\frac{57}{625}HPNP + \cancel{\frac{171}{1250}HPNN}) \\
 & & \quad +(\frac{57}{625}HNPP + \cancel{\frac{171}{1250}HNPN}) \\
 & & \quad +(\cancel{\frac{171}{1250}HNNP} + \cancel{\frac{513}{2500}HNNN}) \\
(5) & \text{OBSERVE}(& \frac{81}{20000}IPPN + \frac{81}{20000}IPNP + \frac{81}{20000}INPP \\
 & \quad \{\textit{result}_1, \textit{result}_2, \textit{result}_3\} & \quad + \frac{57}{625}HPPN + \frac{57}{625}HPNP \\
 & \quad = \{P, P, N\}) & \quad + \frac{57}{625}HNPP \\
(6) & \text{RETURN}(\textit{individual}) & \frac{243}{20000}I + \frac{171}{625}H \\
 & & \\
 & \text{Validity:} & \frac{243}{20000} + \frac{171}{625} = \frac{1143}{4000} \\
 & \text{Posterior:} & \frac{27}{625}I + \frac{608}{625}H
\end{array}
$$

FIGURE 4. Calculations for the "three unordered Covid tests" problem

negative test and two positive tests in any order than it is to observe exactly the sequence of a positive test, a positive test, and a negative test.

Let us state and solve this variant of the problem. The problem now states that: (1) we pick an individual from the population; (2,3,4) we perform three tests; (5) we observe that the multiset induced by these three tests is $\{P, P, N\}$; (6) we want to know the posterior of an illness on the patient.

4. PROBLEM – MONTY FALL

In this variant, once you have selected one of the three doors, the host slips on a banana peel and accidentally pushes open another door, which just happens not to contain the car. Now what are the probabilities that you will win the car if you stick with your original selection, versus if you switch to the remaining door?

Let us solve this version. The problem now states that: (1) the car may be in any of the doors, following a uniform distribution; (2) we observe it is not on the left door; and (3) we want to know where it is. Now the assumption of a host opening doors, that knows where the car is disappears: if some people distrusted vos Savant's solution of the problem [vS], their only argument left is that were solving this easier problem instead.

5. PROBLEM – NEWCOMB'S PARADOX

Suppose a being in whose power to predict your choices you have enormous confidence. There are two boxes, (Bl) and (B2). (Bl) contains \$1000. (B2) contains either \$1000000 (\$M), or nothing.

(1)	$car \leftarrow \textsc{uniform}\{L, M, R\}$	$\frac{1}{3}(\cancel{L} + M + R)$
(2)	$\textsc{observe}(car \neq L)$	$\frac{1}{3}(M + R)$
(3)	$\textsc{return}(car)$	$\frac{1}{3}M + \frac{1}{3}R$
	Validity:	$\frac{2}{3}$
	Posterior:	$\frac{1}{2}M + \frac{1}{2}R$

> *You have a choice between two actions: (1) taking what is in both boxes; or (2) taking only what is in the second box. (1) If the being predicts you will take what is in both boxes, he does not put the \$M in the second box. (2) If the being predicts you will take only what is in the second box, he does put the \$M in the second box.*
>
> *The situation is as follows. First the being makes its prediction. Then it puts the \$M in the second box, or does not, depending upon what it has predicted. Then you make your choice. What do you do?* — Robert Nozick, "Newcomb's problem and two principles of choice" [Noz69]

Let us solve Newcomb's problem in Figure 5. We may take two actions: *one-boxing* (O), and *two-boxing* (T). The problem states that: (1) agents one-box or two-box, we assume uniformly; (2) predictors may declare one-boxing or two-boxing uniformly; (3) we observe that the predictor is accurate; (4) and we compute the outcome according to a payoff matrix. Evidential decision theory prescribes that the best action is the action that, when observed, maximizes the outcome: (5a,6a) *one-boxing* leads to \$1000; (5b,6b) *two-boxing* leads to \$1.

(1)	$action \leftarrow \textsc{uniform}\{O, T\}$	$\frac{1}{2}O + \frac{1}{2}T$
(2)	$prediction \leftarrow \textsc{uniform}\{O, T\}$	$\frac{1}{2}O(\frac{1}{2}O + \cancel{\frac{1}{2}T}) + \frac{1}{2}T(\cancel{\frac{1}{2}O} + \frac{1}{2}T)$
(3)	$\textsc{observe}(action = prediction)$	$\frac{1}{4}OO + \frac{1}{4}TT$
(4)	$outcome \leftarrow \textsc{case}\ (action, prediction)\ \textsc{of}$	$\frac{1}{4}OO(1000) + \frac{1}{4}TT(1)$
	$(O, O) \mapsto 1000$	
	$(O, T) \mapsto 0$	
	$(T, O) \mapsto 1001$	
	$(T, T) \mapsto 1$	
(5a)	$\textsc{observe}(action = O)$	$\frac{1}{4}OO(1000)$
(6a)	$\textsc{return}(outcome)$	1000
(5b)	$\textsc{observe}(action = T)$	$\frac{1}{4}TT(1)$
(6b)	$\textsc{return}(outcome)$	1

FIGURE 5. Solution of the Newcomb's problem.

6. PROBLEM – DEATH IN DAMASCUS

> *Consider the story of the man who met death in Damascus. Death looked surprised, but then recovered his ghastly composure and said, "I am coming for you tomorrow." The terrified man that night bought a camel and rode to Aleppo. The next day, death knocked on the door of the room where he was hiding and said "I have come for*

you." "But I thought you would be looking for me in Damascus." said the man.

"Not at all," said death "that is why I was surprised to see you yesterday. I knew that today I was to find you in Aleppo."

Now suppose the man knows the following. Death works from an appointment book which states time and place; a person dies if and only if the book correctly states in what city he will be at the stated time. The book is made up weeks in advance on the basis of highly reliable predictions. An appointment on the next day has been inscribed for him. Suppose, on this basis, the man would take his being in Damascus the next day as strong evidence that his appointment with death is in Damascus, and would take his being in Aleppo the next day as strong evidence that his appointment is in Aleppo.

Let us solve the problem in Figure 6. The problem assumes that: (1) the merchant could pick any strategy – Fleeing, Staying, or throwing a Random coin – and we assume the uniform distribution as a prior; (2) Death may go to Aleppo or Damascus according to its accurate prediction of what the merchant does; in particular, Death can predict the use of a coin, but not its outcome; (3) the final location of the merchant is given by its strategy and – when relevant – the output of the coin; (4) the outcome is computed from the location of the merchant and Death. Evidential decision theory prescribes the action we would like to observe: (5a) staying and (5b) fleeing both cause the merchant to find Death; but (5c) throwing a random coin allows for some probability of surviving.

(1)	$\mathit{merchant} \leftarrow \textsc{uniform}\{F, S, R\}$	$\frac{1}{3}F + \frac{1}{3}S + \frac{1}{3}R$
(2)	$\mathit{death} \leftarrow \textsc{case}\ \mathit{merchant}\ \textsc{of}$ $F \mapsto A;\quad S \mapsto D;$ $R \mapsto \textsc{uniform}\{A, D\};$	$\frac{1}{3}FA + \frac{1}{3}SD + \frac{1}{3}R(\frac{1}{2}A + \frac{1}{2}D)$
(3)	$\mathit{location} \leftarrow \textsc{case}\ \mathit{merchant}, \mathit{coin}\ \textsc{of}$ $F \mapsto A;\quad S \mapsto D;$ $R \mapsto \textsc{uniform}\{A, D\};$	$\frac{1}{3}FAA + \frac{1}{3}SDD + \frac{1}{6}RA(\frac{1}{2}A + \frac{1}{2}D)$ $+\frac{1}{6}RD(\frac{1}{2}A + \frac{1}{2}D)$
(4)	$\mathit{outcome} \leftarrow \textsc{case}\ \mathit{death}, \mathit{location}\ \textsc{of}$ $A, A \mapsto \textsc{Death};\quad D, D \mapsto \textsc{Death};$ $A, D \mapsto \textsc{Live};\quad D, A \mapsto \textsc{Live};$	$\frac{1}{3}F\textsc{Death} + \frac{1}{3}S\textsc{Death}$ $+\frac{1}{12}R\textsc{Live} + \frac{1}{12}R\textsc{Death}$ $+\frac{1}{12}R\textsc{Live} + \frac{1}{12}R\textsc{Death}$
(5a)	$\textsc{observe}(\mathit{merchant} = S)$	$\frac{1}{3}S\textsc{Death}$
(5b)	$\textsc{observe}(\mathit{merchant} = F)$	$\frac{1}{3}F\textsc{Death}$
(5c)	$\textsc{observe}(\mathit{merchant} = R)$	$\frac{1}{6}R\textsc{Death} + \frac{1}{6}R\textsc{Live}$

FIGURE 6. Solving Death in Damascus.

7. Problem – The Three Prisoners

Three prisoners A, B, and C, are on death row. The governor decides to pardon one of the three and chooses at random the prisoner to pardon. He informs the warden of his choice but requests that the name be kept secret for a few days. The next day, A tries to get the warden to tell him who had been pardoned. The warden refuses.

> *A then asks which of B or C will be executed. The warden thinks for a while, then tells A that B is to be executed.*

Let us write and solve the "Three Prisoners" problem. The problem states that: (1) the governor chooses at random the prisoner to pardon; (2) the warden is informed of the pardoned prisoner, when asked, the warden must not disclose the secret: if A is to be executed, he cannot say so, but if B or C are to be executed, then he can say so; (3) we observe that the warden says B; and (4) we want to know what the governor said.

$$
\begin{array}{lll}
(1) & \mathit{governor} \leftarrow \textsc{uniform}\{A, B, C\} & \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C \\
(2) & \mathit{warden} \leftarrow \textsc{case}\ \mathit{governor}\ \textsc{of} & \frac{1}{3}A(\frac{1}{2}B + \cancel{\frac{1}{2}C}) + \frac{1}{3}BB + \cancel{\frac{1}{3}CC} \\
 & \quad A \mapsto \textsc{uniform}\{B, C\}; & \\
 & \quad B \mapsto \textsc{uniform}\{B\}; & \\
 & \quad C \mapsto \textsc{uniform}\{C\}; & \\
(3) & \textsc{observe}(\mathit{warden} = B) & \frac{1}{6}AB + \frac{1}{3}BB \\
(4) & \textsc{return}(\mathit{governor}) & \frac{1}{6}A + \frac{1}{3}B \\
 & & \\
 & \text{Validity:} & \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \\
 & \text{Posterior:} & \frac{1}{3}A + \frac{2}{3}B
\end{array}
$$

FIGURE 7. Calculations for the "Three Prisoners" problem.

8. Problem – Sailor's Child

> *A Sailor sails regularly between two ports, in each of which he stays with a woman, both of whom wish to have a child by him. He is reluctant, but eventually decides that he will have one or two children, with the number decided by a coin toss – one if Heads, two if Tails.*
>
> *Furthermore, he decides that if the coin lands Heads, he will have a child with the woman who lives in the city listed first in The Sailor's Guide to Ports. (He considers this fair, since although he owns a copy of this book, he hasn't previously read it, and so has no prior knowledge of which city comes first.) Now, suppose that you are this Sailor's child, and that neither you nor your mother know whether he had a child with the other woman. You also do not have a copy of The Sailor's Guide to Ports. You do, however, know that he decided these matters as described above. What should you consider to be the probability that you are his only child (i.e., that the coin he tossed landed Heads)?*
>
> — Radford Neal, "Puzzles of anthropic reasoning resolved using full non-indexical conditioning" [Nea06].

Let us solve the "Sailor's child problem". The problem states that: (1) the Sailor's Guide to ports may list—say—Siracuse (S) or Heraklion (K) as its first port, we assume a uniform distribution; (2) we throw a coin that can land heads (H) or tails (T); (3) we observe that Siracuse is among the cities decided by the Sailor; and (4) we want to know how the coin landed. The solution is in Figure 8.

$$
\begin{array}{lll}
(1) & \textit{guide} \leftarrow \textsc{uniform}\{S, K\} & \frac{1}{2}S + \frac{1}{2}K \\
(2) & \textit{coin} \leftarrow \textsc{uniform}\{H, T\} & \frac{1}{2}S(\frac{1}{2}H + \frac{1}{2}T) + \frac{1}{2}K(\cancel{\frac{1}{2}H} + \frac{1}{2}T) \\
(3) & \textsc{observe}(S \in \textsc{case}\ (\textit{guide}, \textit{coin})\ \textsc{of} & \frac{1}{4}SH + \frac{1}{4}ST + \frac{1}{4}KT \\
 & \quad (S,T) \mapsto \{S,H\};\ (S,H) \mapsto \{S\}; & \\
 & \quad (K,T) \mapsto \{S,H\};\ (K,H) \mapsto \{K\}; & \\
(4) & \textsc{return}(\textit{coin}) & \frac{1}{4}H + \frac{1}{2}T \\
 & & \\
 & \text{Validity:} & \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \\
 & \text{Posterior:} & \frac{1}{3}H + \frac{2}{3}T
\end{array}
$$

FIGURE 8. Solution of the Sailor's Child problem

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