Profunctor optics, a categorical update

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Joint work with


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Emily Pillmore

Thanks go to the organizers of the ACT School.

Part 0: Motivation

(1982) Oles, *"A category-theoretic approach to the semantics of programming languages"*. (2007) Palmer, *"Making Haskell nicer for game programming"*. (2009) Van Laarhoven, *"CPS based functional references"*. (2012) Kmett, *"Lens library"*. (2017) Milewski, *"Profunctor optics: The Categorical View"*. (2017) Pickering, Gibbons, Wu. *"Profunctor optics: modular data accessors"* (2018) Boisseau, Gibbons, *"What You Needa Know About Yoneda"*. (2018) Riley, *"Categories of optics"*.

Optics

Optics are composable bidirectional data accessors. They allow us to access and modify nested data structures.

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- Lenses access specific parts of a data structure.
- Prisms pattern match.
- Traversals iterate over containers.

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- Prisms pattern match.
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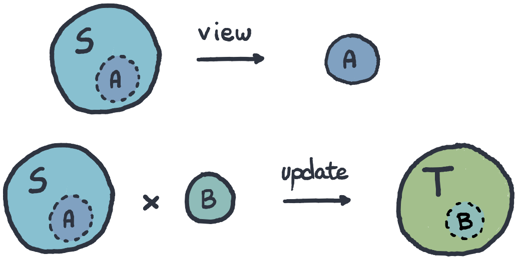
Two optics (of any two families!) can be directly composed.

Lens

Definition

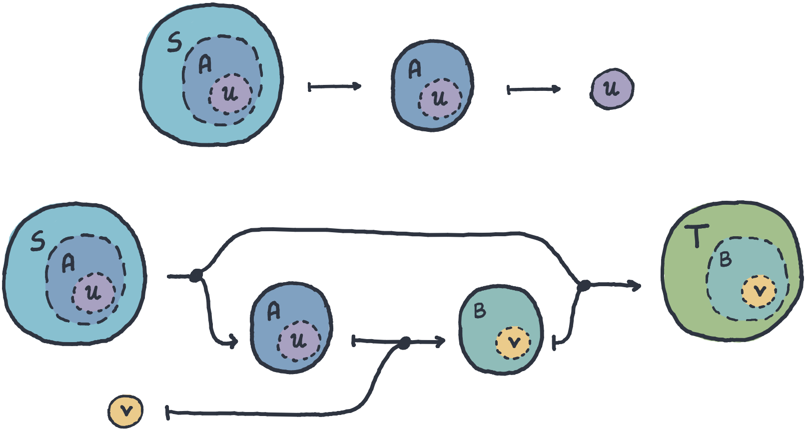
$$\textbf{Lens}\,(A, B; S, T) = \hom(S, A) \times \hom(S \times B, T).$$




```haskell
data Lens a b s t = Lens
  { view :: s → a
  , update :: (s , b) → t }
```

Lenses form a category

Composing $(S, T) \to (A, B) \to (U, V)$.



This preformal intuition can be made into a diagram in a monoidal category.

Example: Lenses

Lenses compose.

```
sherlock = Person
  { name' = "Sherlock Holmes"
  , home' = Address
     { street' = "221b Baker Street"
     , city' = "London"
     , country' = "UK" }
  , occupation = "Consulting detective" }
```

```
>>> view (home) sherlock
```

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>>> view (home.street) sherlock
```

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```

```
>>> view (home.street) sherlock
"221b Baker Street"
```

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```

```
>>> update (home.street) sherlock "4 Marylebone Road"
```

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```
>>> update (home.street) sherlock "4 Marylebone Road"
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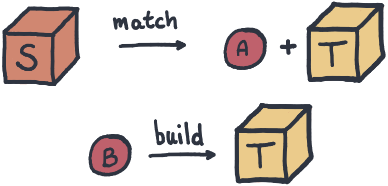
  , home' = Address
     { street' = "4 Marylebone Road"
     , city' = "London"
     , country' = "UK" }
  , occupation = "Consulting detective" }
```

Prisms

Definition

$$\textbf{Prism}\,(A, B; S, T) = \text{hom}(S, T + A) \times \text{hom}(B, T).$$

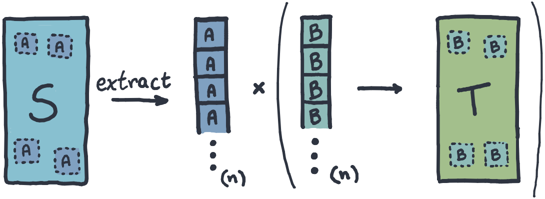



```haskell
data Prism a b s t = Prism
  { match :: s → Either a t
  , build :: b → t }
```

Traversals

Definition

$$\textbf{Traversal}\,(A, B; S, T) = \hom\left(S, \sum_{n\in\mathbb{N}} A^n \times [B^n, T]\right).$$




```
data Traversal a b s t = Traversal
  { extract :: s → ([a] , [b] → t) }
```

The problem of modularity

- How to compose any two optics?
- Even from different families of optics (lens+prism+traversal).
- Simple but tedious code.
- Every pair of families needs special attention.

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- How to compose any two optics?
- Even from different families of optics (lens+prism+traversal).
- Simple but tedious code.
- Every pair of families needs special attention.

Solution: There is an alternative representation in terms of profunctors, which makes composition much easier.

$$\Big(\mathbf{A}(S, A) \times \mathbf{A}(S \times B, T)\Big) \cong \int_{P \in \mathbf{Tambara}} \mathbf{Set}(P(A, B), P(S, T))$$

```
Lens a b s t ≅ (forall p . Tambara p ⇒ p a b → p s t)
```

Where Tambara modules are an algebraic structure studied for the convolution centre of monoidal categories.

Example: Lenses

Lenses compose with ordinary function composition.

```
sherlock = Person
  { name' = "Sherlock Holmes"
  , home' = Address
     { street' = "221b Baker Street"
     , city' = "London"
     , country' = "UK" }
  , occupation = "Consulting detective" }
```

```
>>> update (home.street) sherlock "4 Marylebone Road"
Person
  { name' = "Sherlock Holmes"
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```

Overview

- A general unified definition **Optic** that models all the existing ones and leads to new ones.
- A general representation theorem for any optic in terms of Tambara modules.

$$\mathbf{Optic}(A, B; S, T) \cong \int_{P \in \mathbf{Tambara}} \mathbf{Set}(P(A, B), P(S, T))$$

- Along the way, a bit of coend calculus.

Part 1: Coend Calculus

(1978) MacLane, *"Categories for the working mathematician"* (§IX) (2001) Cáccamo, Winskel, *"A higher-order calculus for categories"*. (2015) Loregian, *"This is the (co)end, my only (co)friend"*.

Ends

Ends are certain kinds of limits for $P\colon \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Set}$.

$$\prod_{X \in \mathbf{A}} P(X, X) \xrightarrow{\pi_A} P(A, A)$$

$$\prod_{X \in \mathbf{A}} P(X, X) \xrightarrow{\pi_B} P(B, B)$$

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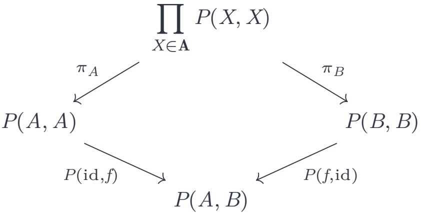

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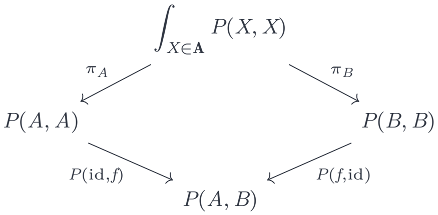

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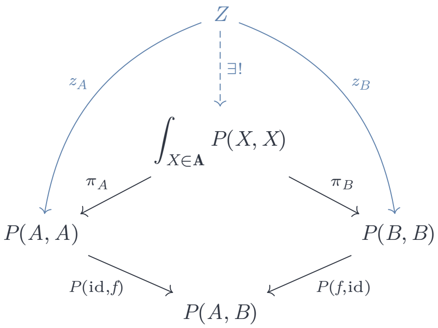

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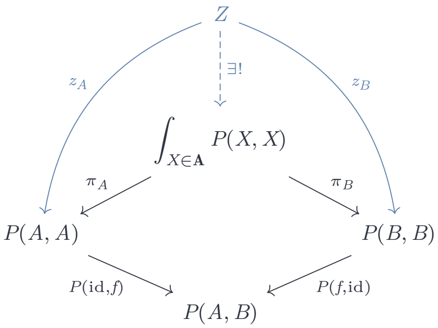

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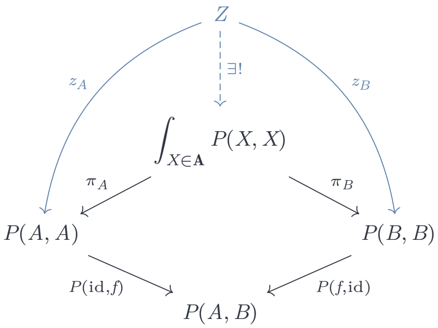


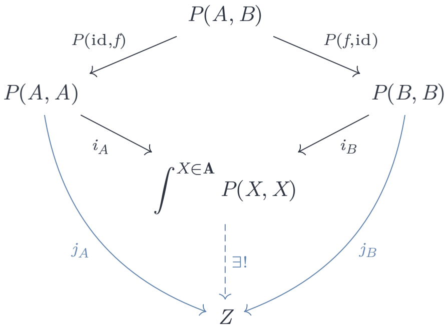
Example

The set of natural transformations can be described as an end.

$$\mathrm{Nat}(F, G) := \int_{A \in \mathbf{A}} \mathrm{hom}_{\mathbf{B}}(FA, GA).$$

Coends

Coends are certain kinds of colimits for $P\colon \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Set}$.



Example

Elements of

$$\int^{A \in \mathbf{A}} P(A, A)$$

are pairs $(A, z \in P(A, A))$, quotiented by $P(f, \mathrm{id})(u) \sim P(\mathrm{id}, f)(u)$.

Coend calculus: some rules

- Coyoneda reductions.

$$\int^{X\in\mathbf{A}} \hom_{\mathbf{A}}(X, A) \times FX \cong FA. \qquad \int^{X\in\mathbf{A}} \hom_{\mathbf{A}}(A, X) \times FX \cong FA.$$

- Fubini rule for coends.

$$\int^{X_1\in\mathbf{A}} \int^{X_2\in\mathbf{B}} P(X_1, X_2, X_1, X_2) \cong \int^{X_2\in\mathbf{B}} \int^{X_1\in\mathbf{A}} P(X_1, X_2, X_1, X_2)$$

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Yoneda reduction: proof

$$
\begin{aligned}
& \mathbf{Set}\left(\int^{X\in\mathbf{A}} \hom(A,X) \times FX, B\right) \\
\cong\ & \{\text{Continuity}\} \\
& \int_{X\in\mathbf{A}} \mathbf{Set}\left(\hom(A,X) \times FX, B\right) \\
\cong\ & \{\text{Closed structure}\} \\
& \int_{X\in\mathbf{A}} \mathbf{Set}\left(\hom(A,X), \mathbf{Set}(FX, B)\right) \\
\cong\ & \{\text{Natural transformations are ends}\} \\
& \mathrm{Nat}\left(\hom(A,-), \mathbf{Set}(F(-), B)\right) \\
\cong\ & \{\text{Usual Yoneda lemma}\} \\
& \mathbf{Set}(F(A), B)
\end{aligned}
$$

Following a proof written by Tom Leinster.

Part 2: Optics

(2017) Milewski. (2017) Boisseau, Gibbons. (2018) Riley.

Optics

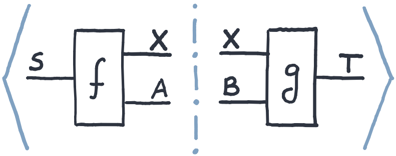
Definition (as in Riley, 2018, Definition 2.0.1)

An optic from (S, T) with *focus* on (A, B) is an element of the following set.

$$\mathbf{Optic}\,(A, B; S, T) := \int^{X \in \mathbf{A}} \hom(S, X \otimes A) \times \hom(X \otimes B, T).$$

Intuition: The optic splits S into some focus A and some *context* X. We cannot access that context, but we can merge it with B to get T.

$$\langle f \colon S \to X \otimes A \mid g \colon X \otimes B \to T \rangle \in \mathbf{Optic}(A, B, S, T)$$



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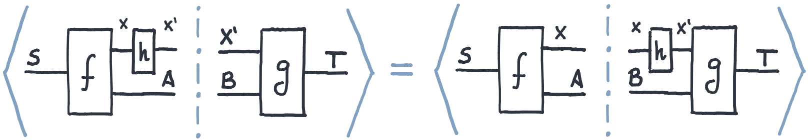
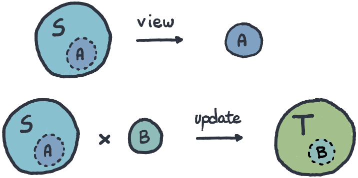
$$\langle f; (h \otimes \mathrm{id}) \mid g \rangle = \langle f \mid (h \otimes \mathrm{id}); g \rangle$$



Lenses are optics



Proposition (as in Milewski, 2017)

Lenses are optics in a cartesian monoidal category.

Proof.

$$
\begin{aligned}
& \int^{X \in \mathbf{A}} \hom(S, X \times A) \times \hom(X \times B, T) \\
\cong\ & \{\text{Adjunction } (\Delta) \dashv (\times)\} \\
& \int^{X \in \mathbf{A}} \hom(S, X) \times \hom(S, A) \times \hom(X \times B, T) \\
\cong\ & \{\text{Yoneda}\} \\
& \hom(S, A) \times \hom(S \times B, T)
\end{aligned}
$$

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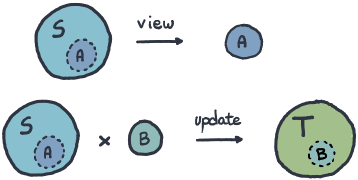

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$$\int^{X \in \mathbf{A}} \hom(S, X \times A) \times \hom(X \times B, T)$$

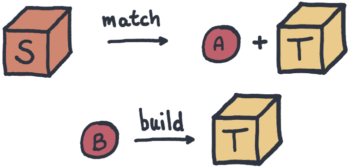
$$\cong \quad \{\text{Adjunction } (\Delta) \dashv (\times)\}$$

$$\int^{X \in \mathbf{A}} \hom(S, X) \times \hom(S, A) \times \hom(X \times B, T) \quad (X = S)$$

$$\cong \quad \{\text{Yoneda}\}$$

$$\hom(S, A) \times \hom(S \times B, T)$$

Prisms are optics



Proposition (Milewski, 2017)

Prisms are optics in a cocartesian monoidal category.

Proof.

$$
\begin{aligned}
& \int^{X \in \mathbf{A}} \hom(S, X + A) \times \hom(X + B, T) \\
\cong \quad & \{\text{Adjunction } (+) \dashv (\Delta)\} \\
& \int^{X \in \mathbf{A}} \hom(S, X + A) \times \hom(X, T) \times \hom(B, T) \\
\cong \quad & \{\text{Yoneda}\} \\
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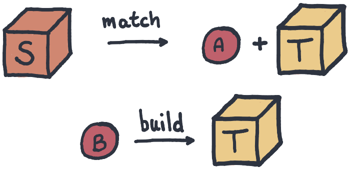
\end{aligned}
$$

Prisms are optics



Proposition (Milewski, 2017)

Prisms are optics in a cocartesian monoidal category.

Proof.

$$\int^{X \in \mathbf{A}} \hom(S, X + A) \times \hom(X + B, T)$$

$\cong$ {Adjunction $(+) \dashv (\Delta)$}

$$\int^{X \in \mathbf{A}} \hom(S, X + A) \times \hom(X, T) \times \hom(B, T) \quad (X = T)$$

$\cong$ {Yoneda}

$$\hom(S, T + A) \times \hom(B, T)$$

Generalizing: actegories, mixed, enriched optics

An optic from $(S, T) \in \mathbf{A}^2$ with *focus* on $(A, B) \in \mathbf{A}^2$ is an element of

$$\mathbf{Optic}\,(A, B; S, T) := \int^{X \in \mathbf{A}} \hom_{\mathbf{A}}(S, X \otimes A) \times \hom_{\mathbf{A}}(X \otimes B, T).$$

1. Basic definition: everything occurs in a monoidal category

$$(\otimes)\colon \mathbf{A} \times \mathbf{A} \to \mathbf{A}.$$

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$$(\otimes) \colon \mathbf{A} \times \mathbf{A} \to \mathbf{A}.$$

2. Monoidal action: an arbitrary $\mathbf{A}$ and a monoidal category $\mathbf{M}$ acting on it

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$$(\otimes)\colon \mathbf{A} \times \mathbf{A} \to \mathbf{A}.$$

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$$(\oslash)\colon \mathbf{M} \times \mathbf{A} \to \mathbf{A}.$$

3. Mixed optic: two categories and two actions

$$(Ⓛ)\colon \mathbf{M} \times \mathbf{A} \to \mathbf{A}, \qquad (Ⓡ)\colon \mathbf{M} \times \mathbf{B} \to \mathbf{B}.$$

Generalizing: actegories, mixed, enriched optics

An optic from $(S, T) \in \mathbf{A}^2$ with *focus* on $(A, B) \in \mathbf{A}^2$ is an element of

$$\mathbf{Optic}\,(A, B; S, T) := \int^{X \in \mathbf{M}} \hom_{\mathbf{A}}(S, X \,Ⓛ\, A) \otimes \hom_{\mathbf{B}}(X \,Ⓡ\, B, T).$$

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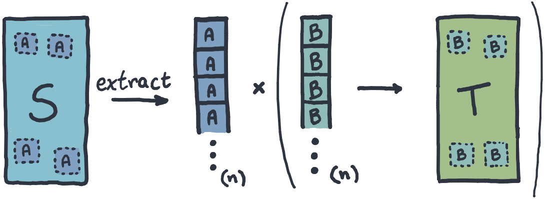
3. Mixed optic: two categories and two actions

$$(Ⓛ) \colon \mathbf{M} \times \mathbf{A} \to \mathbf{A}, \qquad (Ⓡ) \colon \mathbf{M} \times \mathbf{B} \to \mathbf{B}.$$

4. Enriched optics work exactly the same, getting us an *object of optics*. In all of these cases, we get an (enriched) category of optics.

Traversals are optics for power series functors



Proposition

Traversals are optics for power series functors.

$$\int^{X \in [\mathbb{N}, \mathbf{A}]} \hom_{\mathbf{A}} \left(S, \sum_{n \in \mathbb{N}} A^n \otimes X_n \right) \times \hom_{\mathbf{A}} \left(\sum_{n \in \mathbb{N}} B^n \otimes X_n, T \right)$$

$$\cong$$

$$\hom_{\mathbf{A}} \left(S, \sum_{n \in \mathbb{N}} A^n \otimes [B^n, T] \right)$$

Composition monoidal product as defined by Kelly's *"On the Operads of J. P. May"*.

Traversals are optics for power series functors

$$\int^{X \in [\mathbb{N}, \mathbf{A}]} \hom_{\mathbf{A}} \left(S, \sum_{n \in \mathbb{N}} A^n \otimes X_n \right) \times \hom_{\mathbf{A}} \left(\sum_{n \in \mathbb{N}} B^n \otimes X_n, T \right)$$

$\cong$ {Continuity}

$$\int^{X \in [\mathbb{N}, \mathbf{A}]} \hom_{\mathbf{A}} \left(S, \sum_{n \in \mathbb{N}} A^n \otimes X_n \right) \times \prod_{n \in \mathbb{N}} \hom_{\mathbf{A}} \left(X_n \otimes B^n, T \right)$$

$\cong$ {Adjunction $(- \otimes B^n) \dashv [B^n, -]$}

$$\int^{X \in [\mathbb{N}, \mathbf{A}]} \hom_{\mathbf{A}} \left(S, \sum_{n \in \mathbb{N}} A^n \otimes X_n \right) \times \prod_{n \in \mathbb{N}} \hom_{\mathbf{A}} \left(X_n, [B^n, T] \right)$$

$\cong$ {Natural transformation}

$$\int^{X \in [\mathbb{N}, \mathbf{A}]} \hom_{\mathbf{A}} \left(S, \sum_{n \in \mathbb{N}} A^n \otimes X_n \right) \times [\mathbb{N}, \mathbf{A}] \left(X_{(-)}, [B^{(-)}, T] \right)$$

$\cong$ {Coyoneda}

$$\hom_{\mathbf{A}} \left(S, \sum_{n \in \mathbb{N}} A^n \otimes [B^n, T] \right).$$

Proposition

Lenses in a symmetric monoidal category are mixed optics.

$$\int^{X \in \mathbf{A}} \mathbf{Comon}_{\mathbf{C}}(S, X \otimes A) \times \mathbf{C}(\mathcal{U}X \otimes B, T) \cong \mathbf{Comon}_{\mathbf{C}}(S, A) \times \mathbf{C}(S \otimes B, T).$$

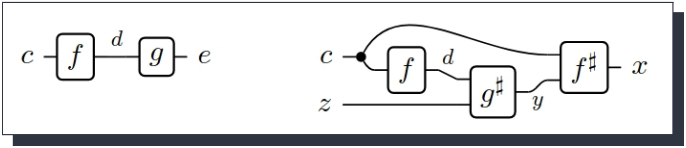


Figure 1: "Generalized lenses via functors $C^{op} \to Cat$", Spivak, (Myers).

Monadic lenses are mixed optics

Proposition

Monadic lenses are mixed optics. For any strong monad $\Psi \colon \mathbf{A} \to \mathbf{A}$,

$$\int_{X \in \mathbf{A}} \mathbf{A}(S, X \times A) \times \mathbf{Kl}_{\Psi}(X \rtimes B, T) \cong \mathbf{A}(S, A) \times \mathbf{A}(S \times B, \Psi T).$$

2.3 Monadic lenses

We propose the following definition of monadic lenses for any monad M:

Definition 2.1 (monadic lens). A *monadic lens* from source type A to view type B in which the put operation may have effects from monad M (or "M-lens from A to B"), is represented by the type $[A \rightsquigarrow B]_M$, where

$$\mathbf{data}\ [\alpha \rightsquigarrow \beta]_{\mu} = \mathit{MLens}\ \{\mathit{mget} :: \alpha \to \beta, \mathit{mput} :: \alpha \to \beta \to \mu\ \alpha\}$$

Figure 2: "Reflections on Monadic lenses", Abou-Saleh, Cheney, Gibbons, McKinna, Stevens.

Table of optics

Name	Actions	From
Adapters	$\mathbf{A}(S, A) \times \mathbf{B}(B, T)$	Kmett, 2012
Setters	$\mathbf{A}(S, A)$	Kmett, 2012
Getters	$\mathbf{A}(B, T)$	Kmett, 2012
Folds	$\mathbf{A}(S, \mathrm{List}(A))$	Kmett, 2012
Lenses	$\mathbf{A}(S, A) \times \mathbf{B}(S \bullet B, T)$	Oles, 1982
Prisms	$\mathbf{A}(S, A \bullet T) \times \mathbf{B}(B, T)$	Kmett, 2012
Grates	$\mathbf{A}(S, [[A, B], T])$	Deikun, O'Connor, 2016
Affine traversal	$\mathbf{A}(S, A \times [B, T] + T)$	Grenrus, 2012
Linear lenses	$\mathbf{A}(S, A \otimes [B, T])$	Riley, 2018
Lenses in a symm. mon.	$\mathbf{Comon}(S, A) \times \mathbf{A}(S \otimes B, T)$	Spivak, Myers, 2019
Monadic lenses	$\mathbf{A}(S, A) \times \mathbf{A}(S \times B, \Psi T))$	Abou-Saleh et al.
Glasses	$\mathbf{A}([[S, A], B], [S, T])$	New
Algebraic lenses	$\mathbf{A}(S, A) \times \mathbf{B}(\Psi S \bullet B, T)$	New
Kaleidoscopes	$\prod_{n \in \mathbb{N}} \mathbf{A}\left([A^n, B], [S^n, T]\right)$	New

Part 4: Tambara modules

(2006) Tambara, (2008) Pastro, Street, (2014) Rivas, Jaskelioff. (2017) Boisseau, Gibbons. (2018) Riley.

Tambara modules

Definition (Pastro and Street, 2008)

Let $\mathbf{A}$ be a monoidal category. A Tambara module is an endoprofunctor $T\colon \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{Set}$ equipped with a family of morphisms natural on both A and B and dinatural on M.

$$t_{A,B,M}\colon \; T(A,B) \to T(M \otimes A, M \otimes B)$$

They come with axioms that make them interplay nicely with the structure isomorphisms of the monoidal category.

There is an adjoint triple, with Ψ an opmonoidal monad and Θ a monoidal comonad.

$$\Theta P(A,B) := \int_{M \in \mathbf{M}} P(M \otimes A, M \otimes B).$$

$$\Psi P(A,B) := \int^{X,Y \in \mathbf{A}, M \in \mathbf{M}} \mathbf{A}(A, M \otimes X) \otimes \mathbf{A}(M \otimes Y, B) \otimes P(X,Y).$$

Finally, $\check{\Psi}$ can be made into a monoid in the bicategory of profunctors. The Kleisli object for it is the category of optics.

Tambara modules (generalized)

Definition (Pastro and Street, 2008)

Let $\mathbf{C}$ and $\mathbf{D}$ be categories with two monoidal actions from $\mathbf{M}$. A Tambara module is an endoprofunctor $T\colon \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Set}$ equipped with a family of morphisms natural on both A and B and dinatural on M.

$$t_{A,B,M}\colon \ T(A,B) \to T(M ⓛ A, M ⓡ B)$$

They come with axioms that make them interplay nicely with the structure isomorphisms of the monoidal actions.

There is an adjoint triple, with Ψ an monad and Θ a comonad.

$$\Psi P(A,B) := \int^{X,Y \in \mathbf{A}, M \in \mathbf{M}} \mathbf{A}(A, M ⓛ X) \otimes \mathbf{B}(M ⓡ Y, B) \otimes P(X,Y).$$

$$\Theta P(A,B) := \int_{M \in \mathbf{M}} P(M ⓛ A, M ⓡ B).$$

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Finally, $\check{\Psi}$ can be made into a monoid in the bicategory of profunctors. The Kleisli object for it is the category of optics.

Lemma

$$[\mathbf{Optic}, \mathbf{Set}] \cong \mathbf{Tambara}$$

Representation theorem

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forall p . (Tambara p) ⇒ p a b → p s t
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Theorem

$$\mathbf{Optic}(A, B, S, T) \cong \int_{P \in \mathbf{Tambara}} \mathbf{Set}(P(A, B), P(S, T)).$$

This is a form of Yoneda. The actual work of the proof is done on characterizing copresheaves as Tambara modules.

$$\mathbf{A}(X, Y) \cong \int_{F \in [\mathbf{A}, \mathbf{Set}]} \mathbf{Set}(FX, FY).$$

In the paper

- More on Tambara theory and the Pastro-Street promonad.
- More examples on how to use optics, both the classical ones and the new ones. They have accompanying code.
- Full derivations for all the optics.
- General mixed, enriched optics.