The Produoidal Algebra of Process Decomposition

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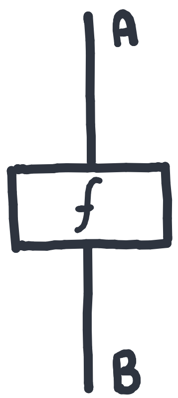
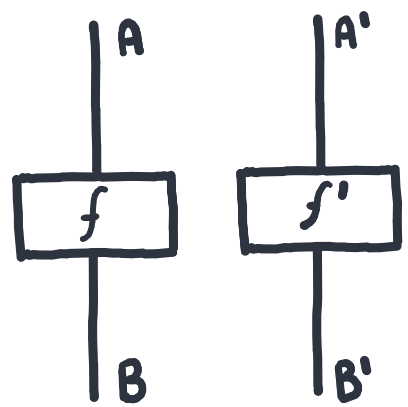
ERC BLAST project.
EU Estonian IT Academy.

MONOIDAL CATEGORIES: PROCESS THEORIES

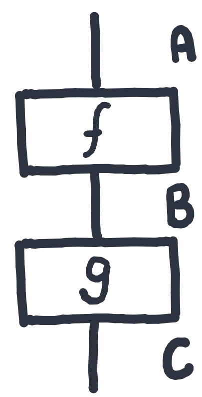
Monoidal categories are an algebra of parallel and sequential composition.
String diagrams are an internal language of monoidal categories.



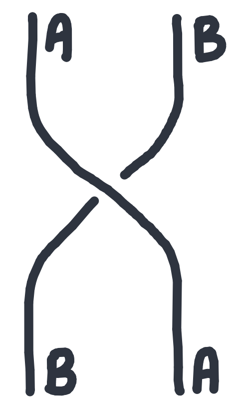
Process



Parallel composition



Sequential composition



Swap

Bènabou

Monoidal Categories: Process Theories

Monoidal categories are an algebra of parallel and sequential composition.
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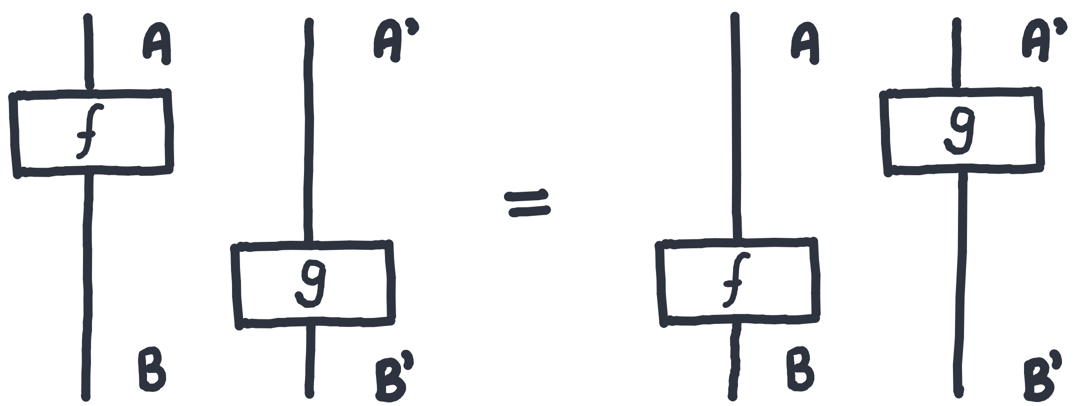
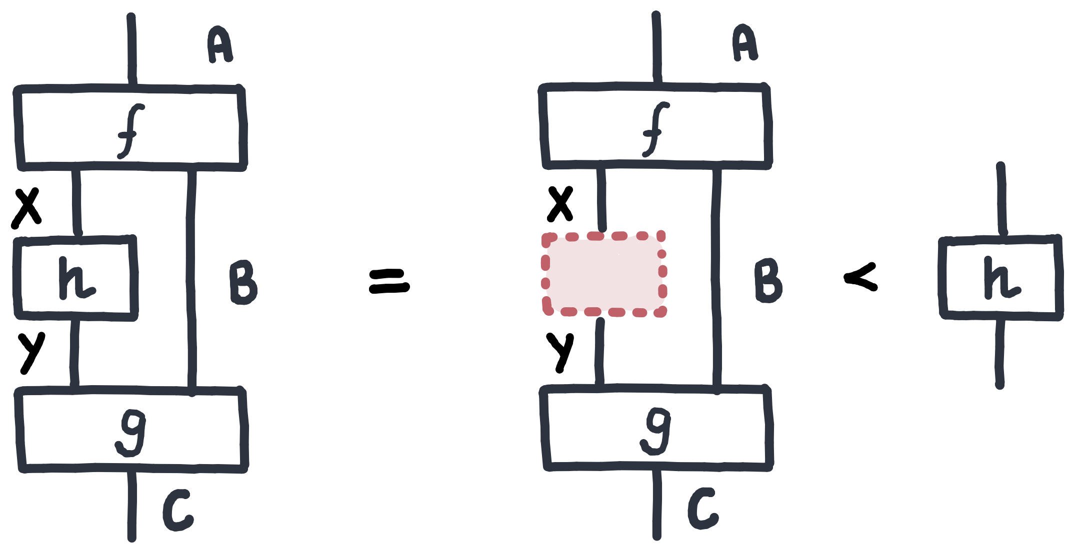


Interchange Law

Bènabou

Monoidal Categories: Process Theories

Monoidal categories are an algebra of parallel and sequential composition.
String diagrams are an internal language of monoidal categories.
But string diagrams do not only decompose sequentially and in parallel.



Bènabou

PART 0: Optics

or Contexts

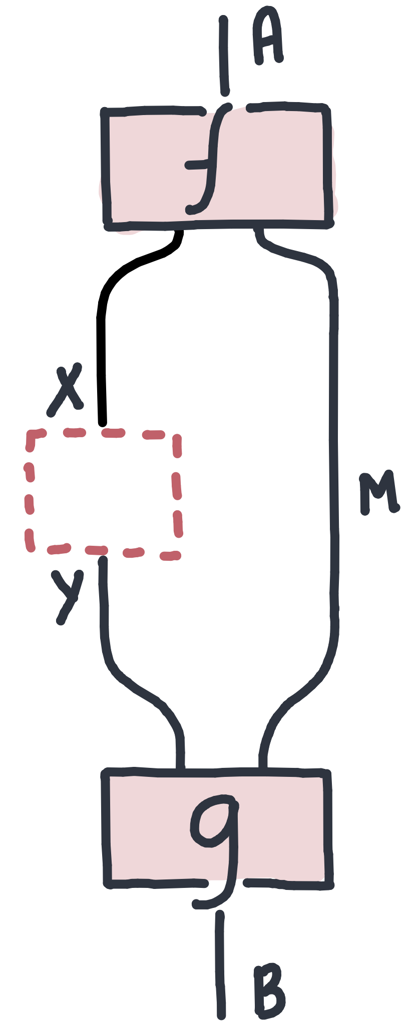
OPTICS

An optic from A to B with a hole from X to Y is a pair of morphisms

$$f : A \to X \otimes M, \qquad g : Y \otimes M \to B,$$

written as $\langle f \mid g \rangle$, and quotiented by dinaturality on M:

$$\langle f ; (id \otimes h) \mid g \rangle = \langle f \mid (id \otimes h) ; g \rangle.$$



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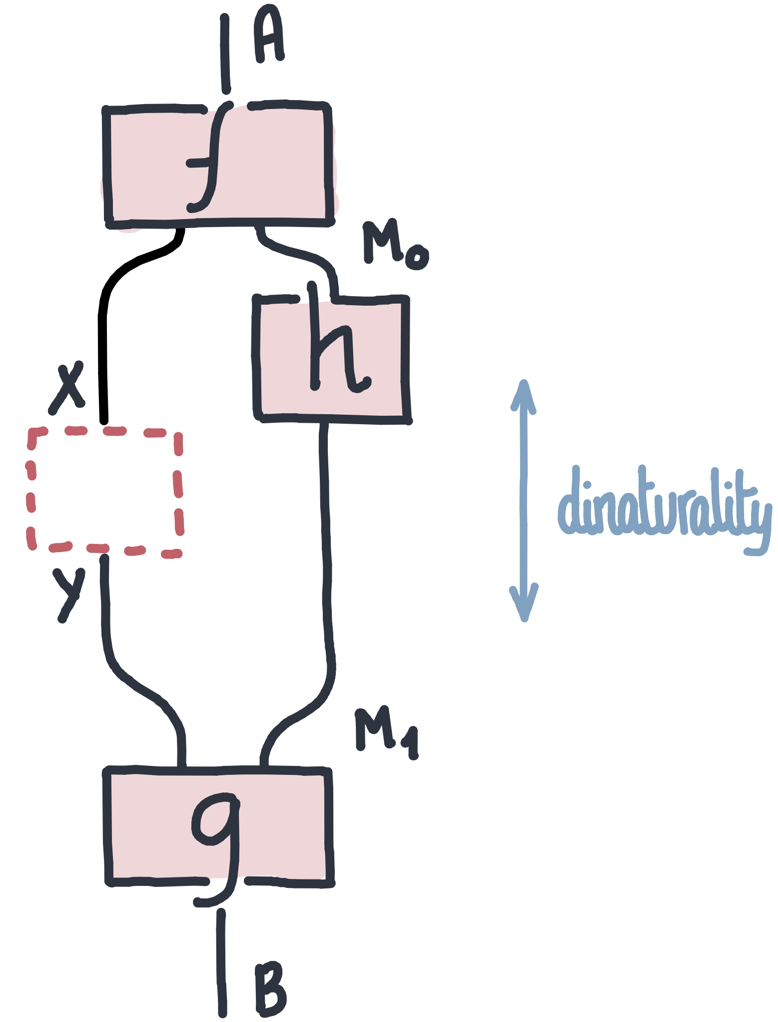

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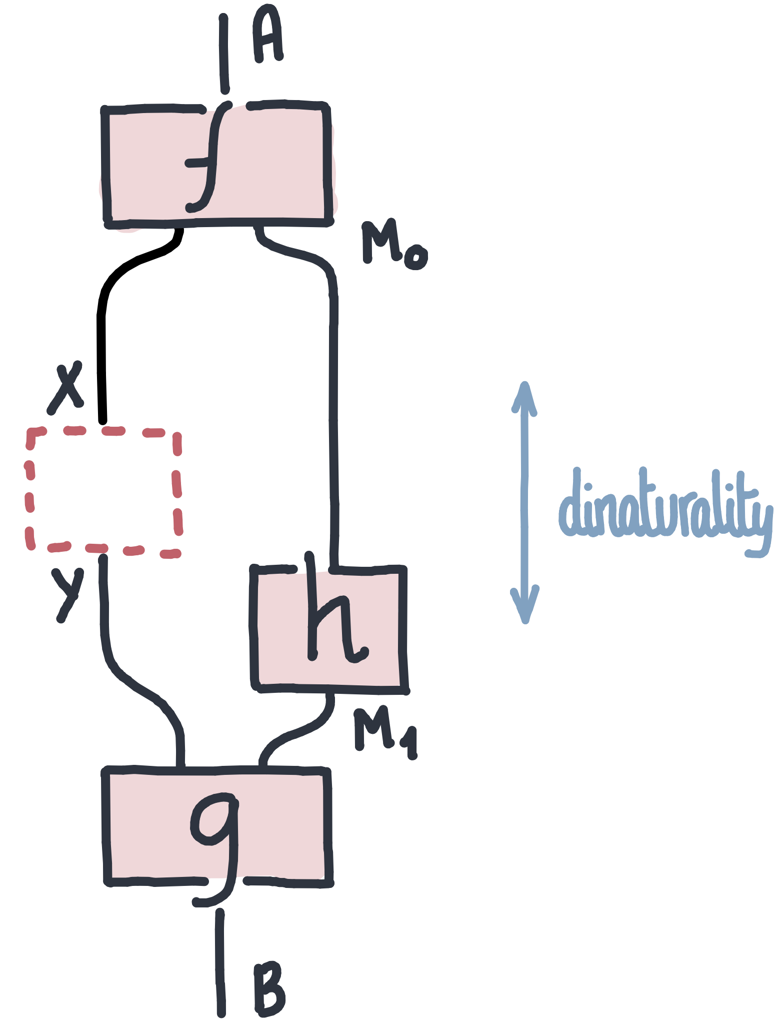
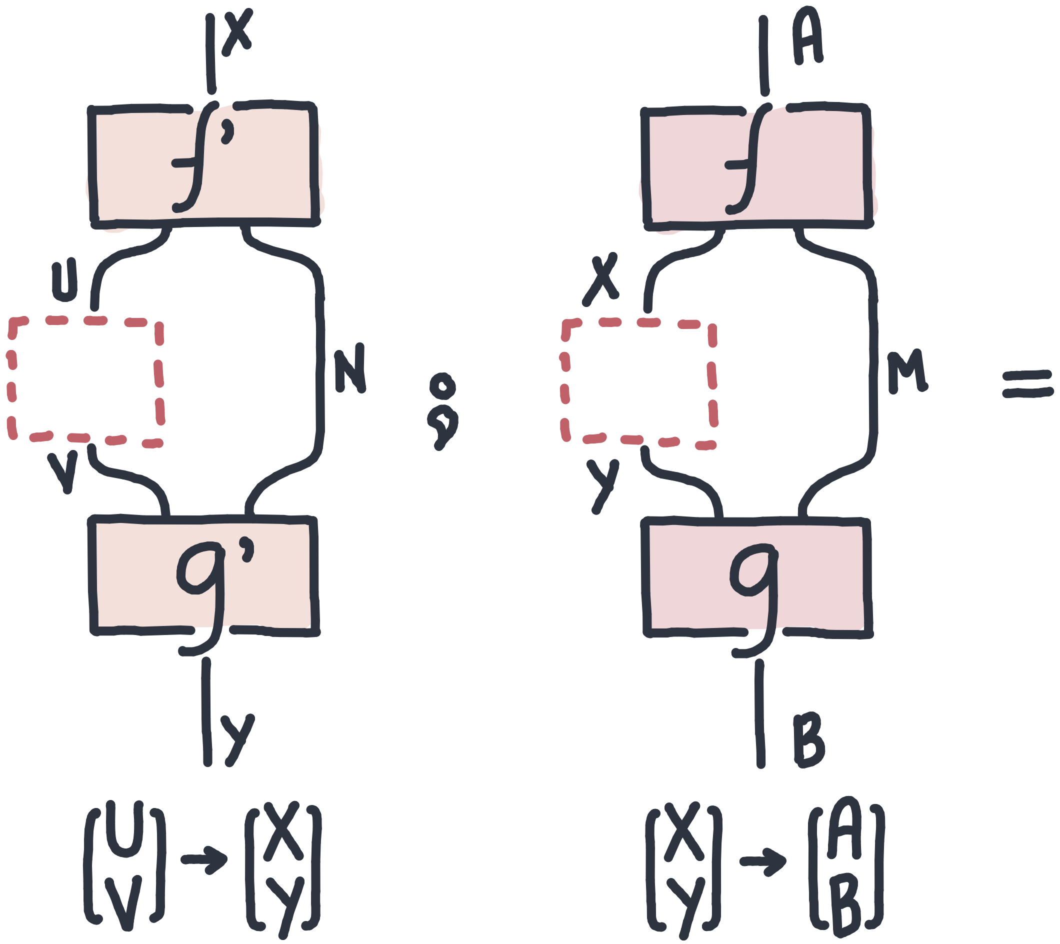
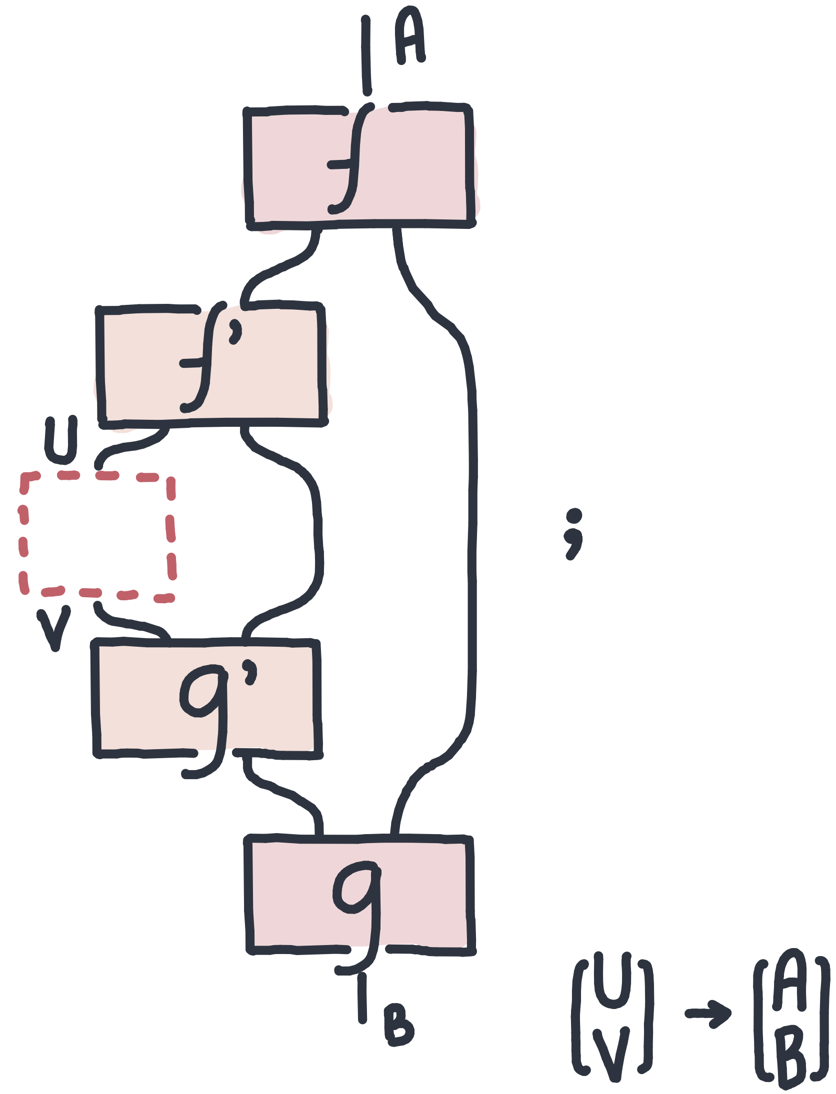


Optics form a Category

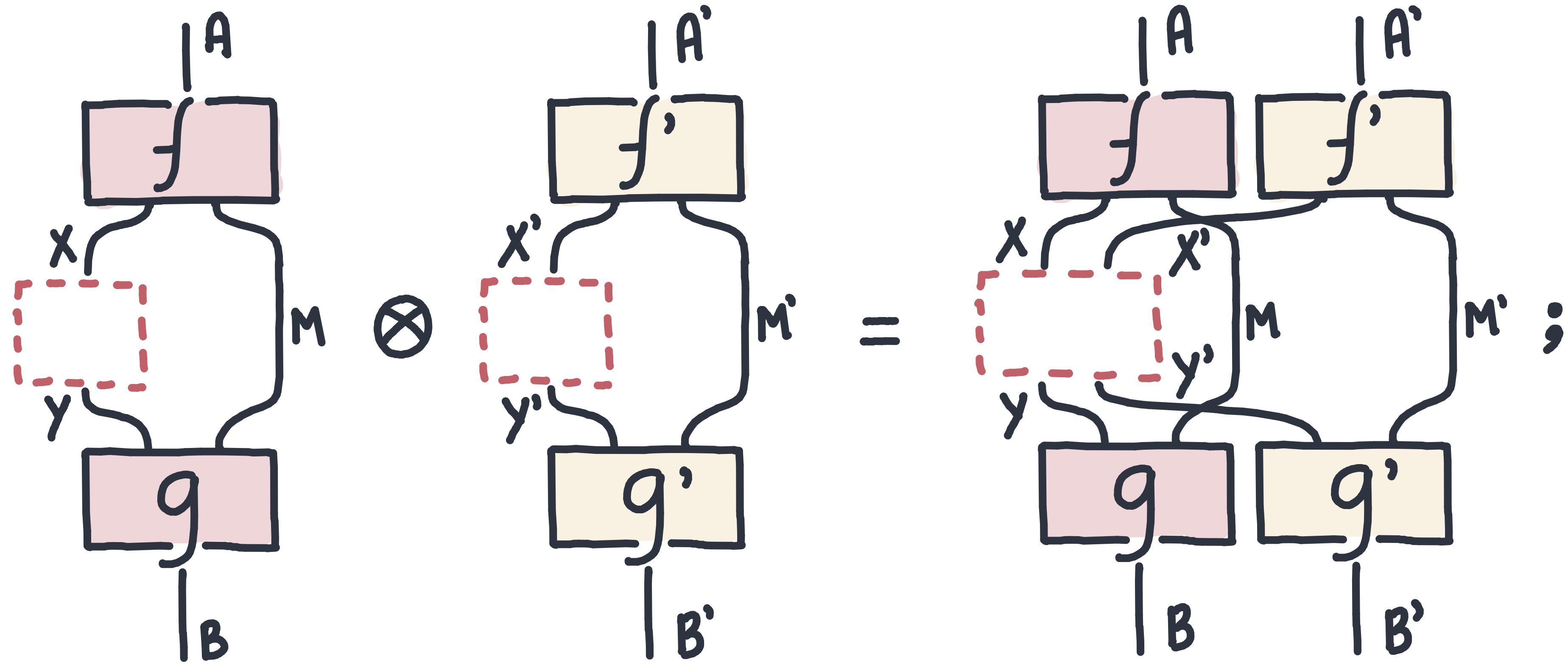
Objects are pairs $\begin{bmatrix} X \\ Y \end{bmatrix}$. Composition is





Optics form a Monoidal Category

Tensoring is $\begin{bmatrix} X \\ Y \end{bmatrix} \otimes \begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} X \otimes X' \\ Y \otimes Y' \end{bmatrix}$, and

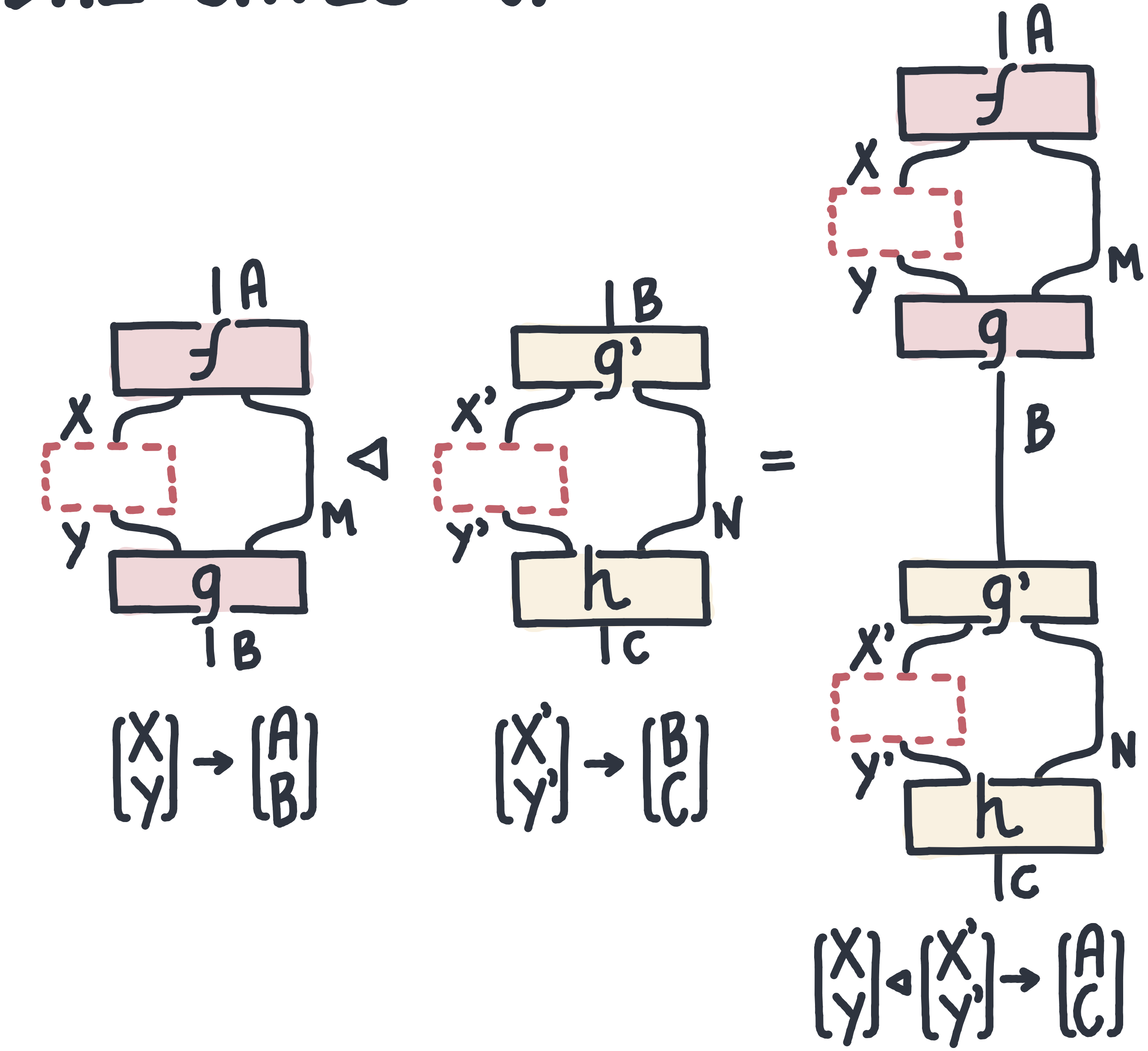


$\begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$ $\otimes$ $\begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} A' \\ B' \end{bmatrix}$ $=$ $\begin{bmatrix} X \otimes X' \\ Y \otimes Y' \end{bmatrix} \to \begin{bmatrix} A \otimes A' \\ B \otimes B' \end{bmatrix}$

Optics form a Duoidal Category?

Sequencing is not an operation: $\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix}$ is not an object, even when $\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ is defined.

This is not monoidal, but it is still promonoidal.



Optics form a Duoidal Category?

Sequencing is not an operation, it defines a hom-set to an object that does not really exist.

$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix}$ is not an object,

but $\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$ is defined.

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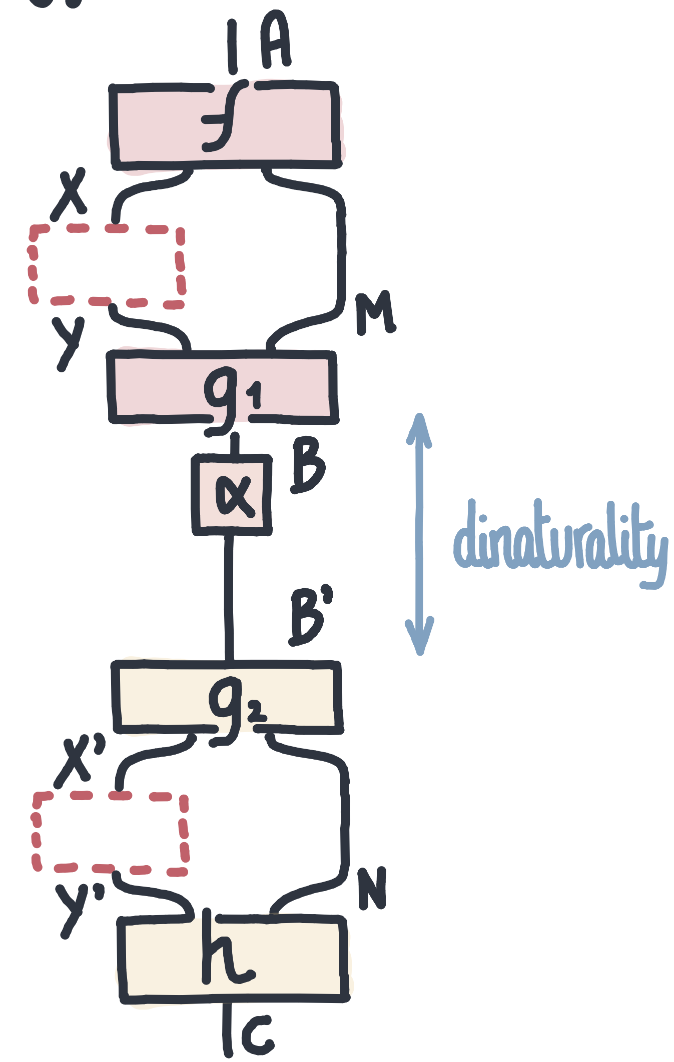

$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ C \end{bmatrix}$

Optics form a Duoidal Category?

Sequencing is not an operation, it defines a hom-set to an object that does not really exist.

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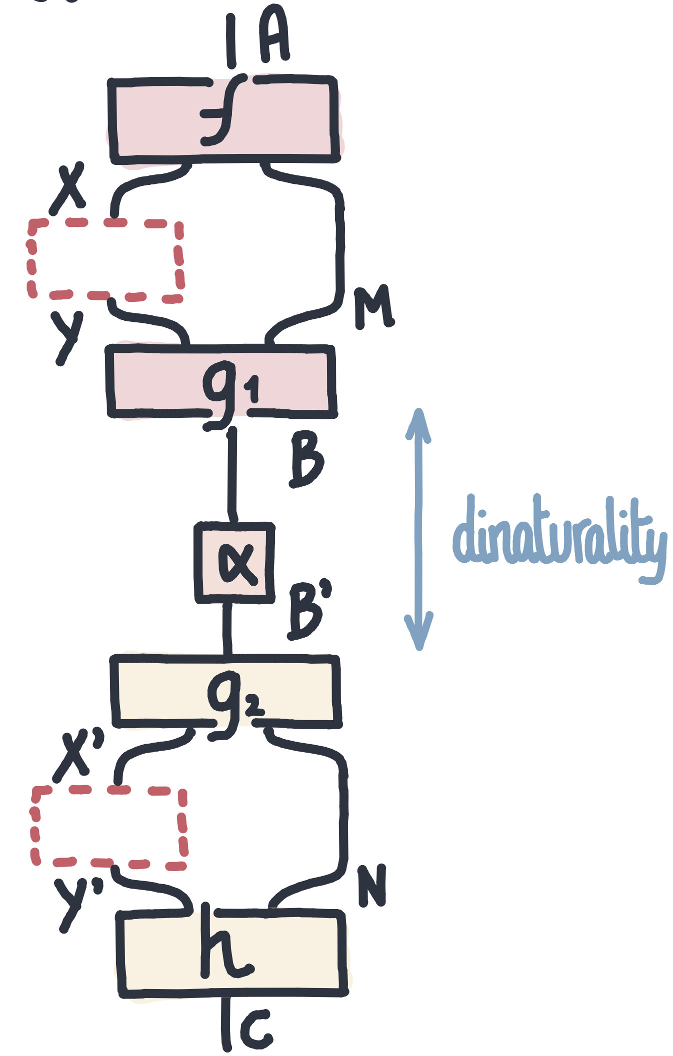

$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$

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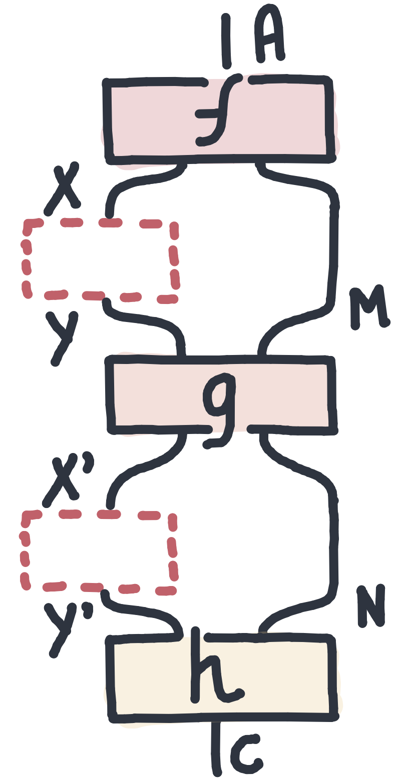


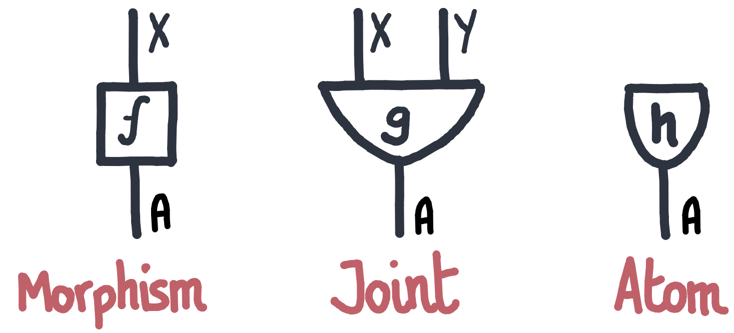
$$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$$

Part 1 : Promonoidals

PROMONOIDAL CATEGORIES

Promonoidal categories provide a theory of coherent composition. It has

- Morphisms, $\mathbb{C}(X;A)$.
- Joints, $\mathbb{C}(X \triangleleft Y;A)$.
- Atoms, $\mathbb{C}(N;A)$.

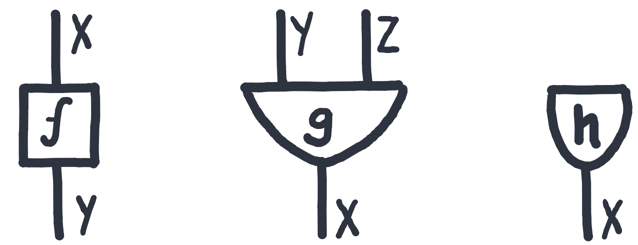


Malleability property: splitting A into X and "something" and then splitting that something into Y and Z can be done in the same number of ways as splitting A into "something" and Z and then splitting that something into X and Y.

ProMonoidal Categories

Promonoidal categories provide a theory of coherent composition. It has

- Morphisms, $\mathbb{C}(X;A)$.
- Joints, $\mathbb{C}(X \triangleleft Y;A)$.
- Atoms, $\mathbb{C}(N;A)$.



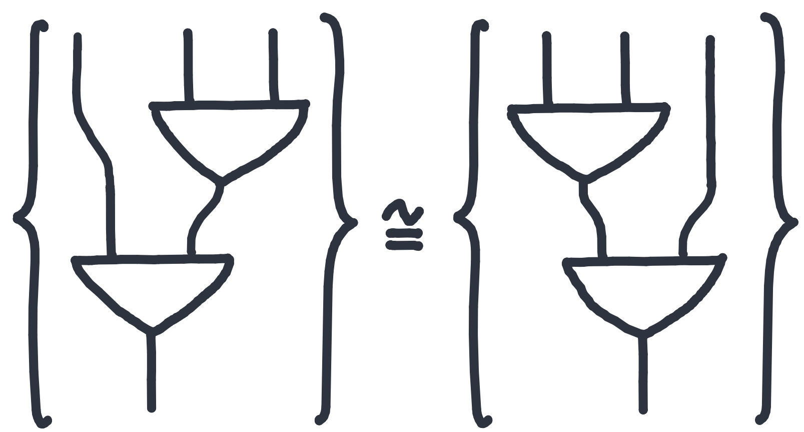
Morphism Joint Atom

Malleability property:

$$\int^{M \in \mathbb{C}} \mathbb{C}(A; X \otimes M) \times \mathbb{C}(M; Y \otimes Z) \cong \int^{M \in \mathbb{C}} \mathbb{C}(A; M \otimes Z) \times \mathbb{C}(M; X \otimes Y);$$

$$\int^{M \in \mathbb{C}} \mathbb{C}(A; X \otimes M) \times \mathbb{C}(M; I) \cong \mathbb{C}(A; X);$$

$$\int^{M \in \mathbb{C}} \mathbb{C}(A; M \otimes X) \times \mathbb{C}(M; I) \cong \mathbb{C}(A; X);$$



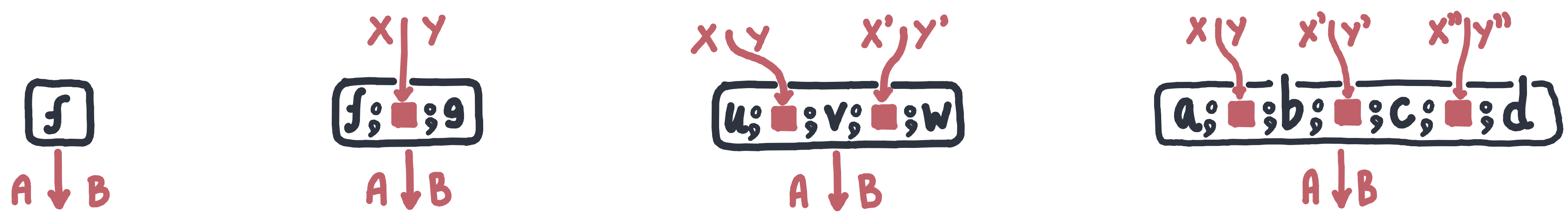
PART 2 : Context for Categories

CONTEXT FOR CATEGORIES

Consider 'expressions with holes' in a category, like the following

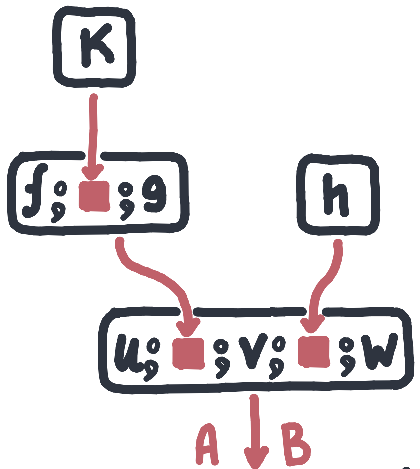
$$u ⨾ \blacksquare ⨾ v ⨾ \blacksquare ⨾ w, \qquad f ⨾ \blacksquare ⨾ g, \qquad f, \qquad a ⨾ \blacksquare ⨾ b ⨾ \blacksquare ⨾ c ⨾ \blacksquare ⨾ d.$$

These contexts form a promonoidal category.



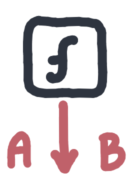
$$\begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \triangleleft \begin{bmatrix}X''\\Y''\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$$

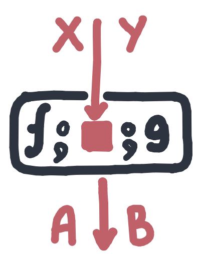
Context for Categories

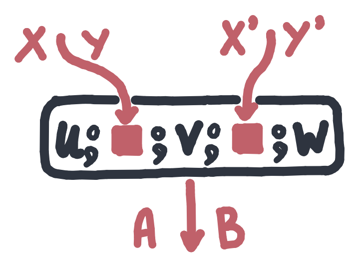


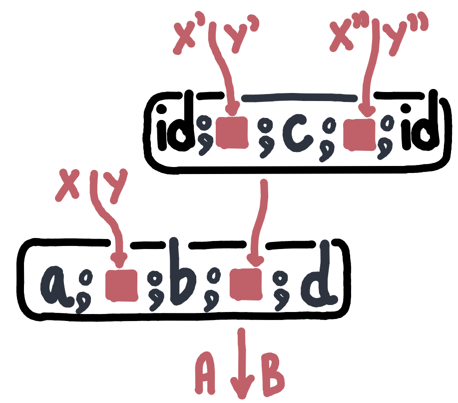
$= u ; f ; k ; g ; v ; h ; w$

These contexts form a promonoidal category.









$\begin{bmatrix}A\\B\end{bmatrix}$

$\begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$

$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$

$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \triangleleft \begin{bmatrix}X''\\Y''\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$

Contour is Adjoint to Splice

What is a canonical algebra of context on top of a category?

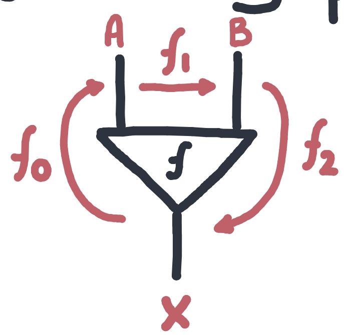
- Each category gives a cofree promonoidal, context.
- Each promonoidal gives a free category, contour.

$$
\begin{tikzcd}
\text{Category} \arrow[rr, bend left, "\text{Context}"] \arrow[rr, phantom, "\top"] & & \text{Promonoidal Category} \arrow[ll, bend left, "\text{Contour}"]
\end{tikzcd}
$$

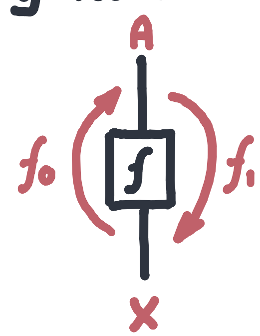
CONTOUR

Melliès, Zeilberger.

Contouring promonoidal categories generates a category.



$f_0 : X^L \to A^L$
$f_1 : A^R \to B^L$
$f_2 : B^R \to X^R$



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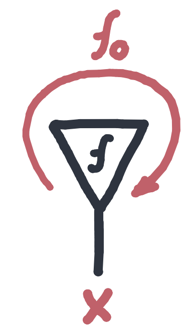
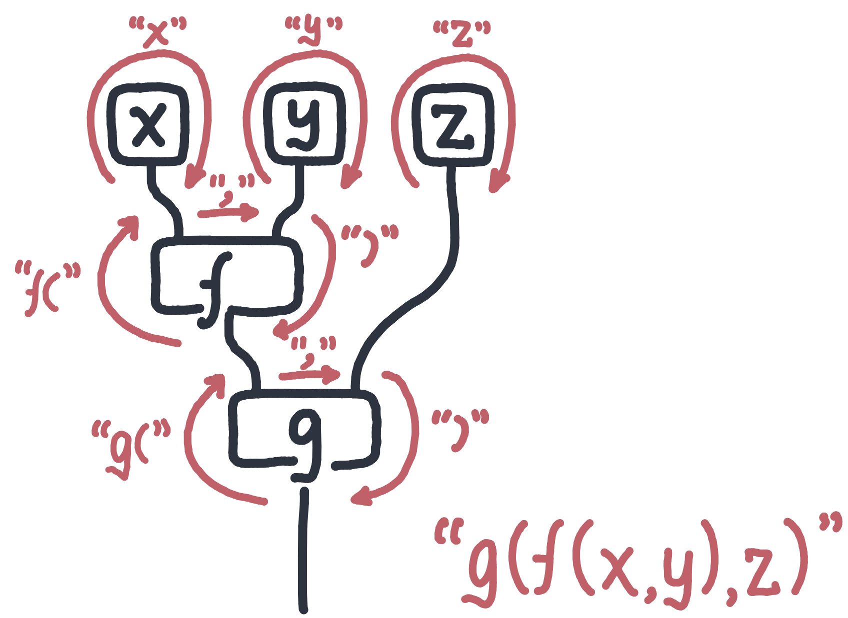


$f_0 : X^L \to X^R.$

The category provides
a simple parsing algebra to
any promonoidal,
or any multicategory.



NEXT

$$
\begin{tikzcd}
\text{Monoidal Category} \arrow[r, bend left, "\text{Mon. Context}"] \arrow[r, phantom, "\perp"] \arrow[d] & \text{Produoidal Category} \arrow[l, bend left, "\text{Mon. Contour}"] \arrow[d] & \\
\text{Category} \arrow[r, bend left, "\text{Context}"] \arrow[r, phantom, "\perp"] & \text{Promonoidal Category} \arrow[l, bend left, "\text{Contour}"] & \text{Duoidal Category} \arrow[ul, dashed]
\end{tikzcd}
$$

Part 3: Context for Monoidal Categories

Produoidal Categories

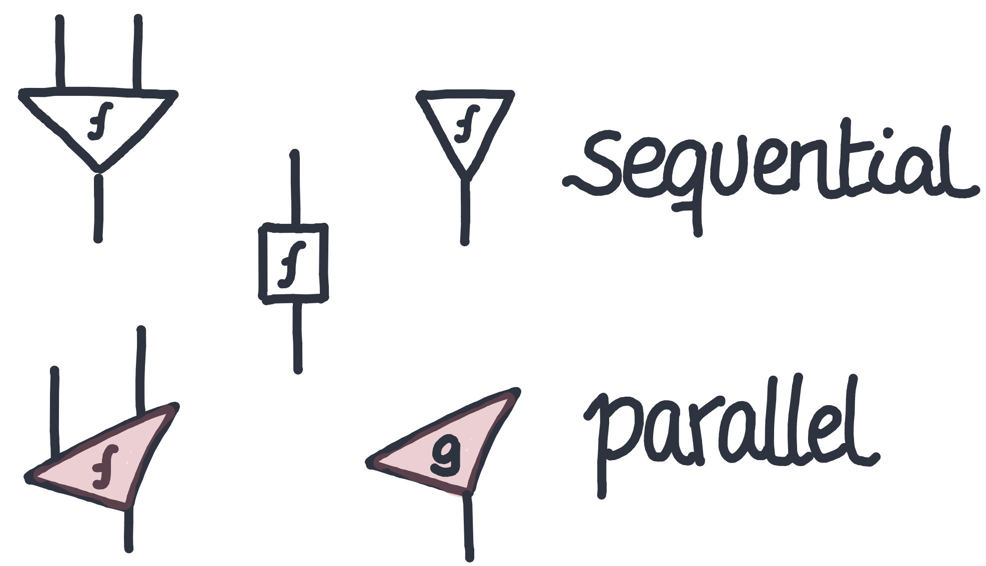
Definition. A produoidal is a pair of promonoidals

$$\mathbb{V}(\bullet \triangleleft \bullet ; \bullet) : \mathbb{V}^{op} \times \mathbb{V}^{op} \times \mathbb{V} \to \mathrm{Set}, \qquad \mathbb{V}(\bullet ; N) : \mathbb{V}^{op} \to \mathrm{Set},$$ "sequential",

$$\mathbb{V}(\bullet ; \bullet \otimes \bullet) : \mathbb{V}^{op} \times \mathbb{V}^{op} \times \mathbb{V} \to \mathrm{Set}, \qquad \mathbb{V}(\bullet ; I) : \mathbb{V}^{op} \to \mathrm{Set},$$ "parallel".

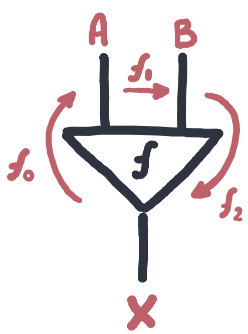
One laxly distributes over the other,

$$\psi_2 : (A \triangleleft B) \otimes (C \triangleleft D) \to (A \otimes C) \triangleleft (B \otimes D),$$
$$\psi_0 : I \to N$$
$$\varphi_2 : N \to N \triangleleft N$$
$$\varphi_0 : I \to I \otimes I$$

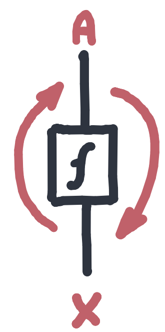


Monoidal Contour

Contouring produoidal categories generates a monoidal category.



$f_0 : X^L \to A^L$
$f_1 : A^R \to B^L$
$f_2 : B^R \to X^R$



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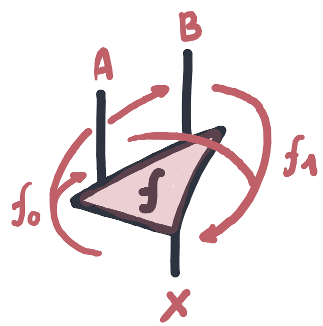
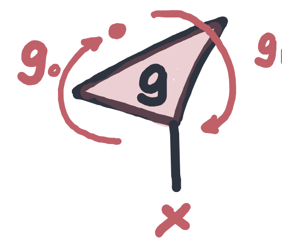


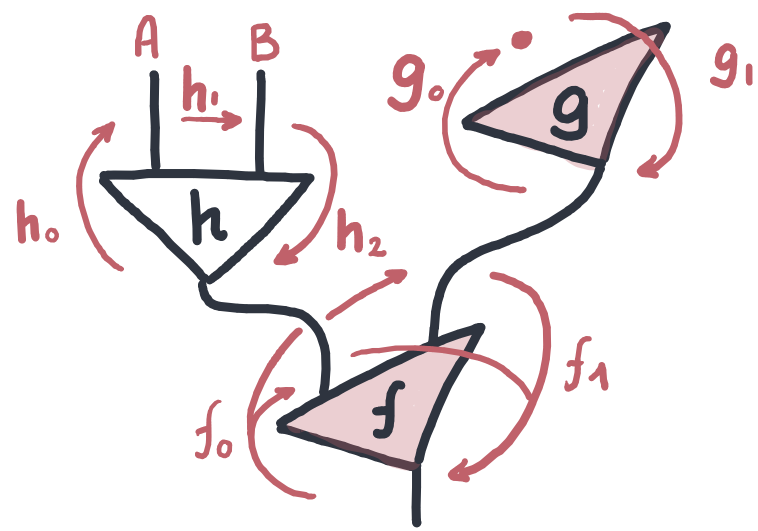
$f_0 : X^L \to A^L \otimes B^L$
$f_1 : A^R \otimes B^R \to X^R$

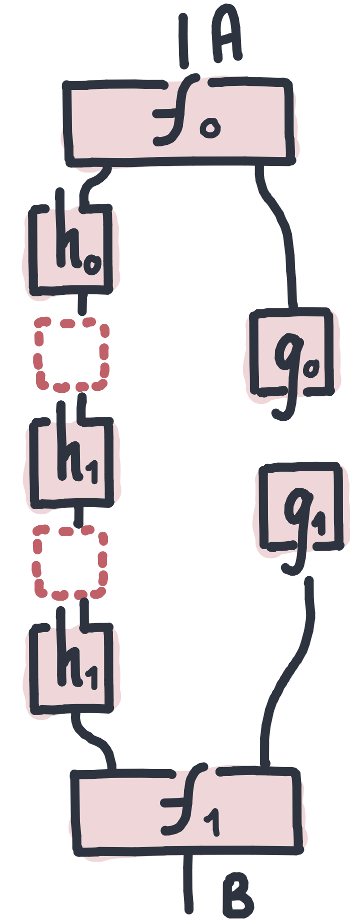


$g_0 : X^L \to I$
$g_1 : I \to X^R$

MONOIDAL CONTOUR

Contouring produoidal categories generates a monoidal category. Example.



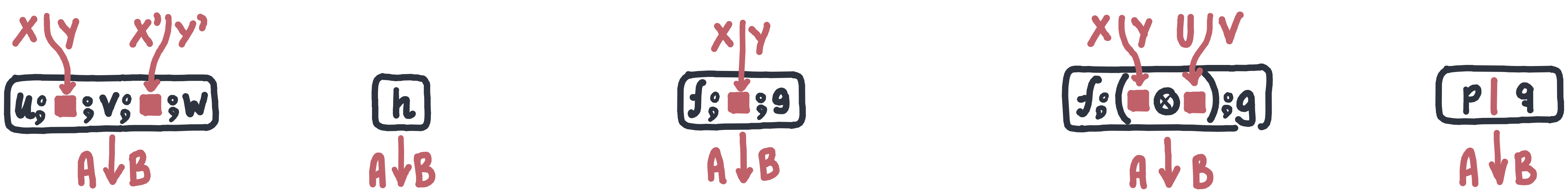


Monoidal Context

Consider 'expressions with holes' in a monoidal category, like the following

$$u ; \blacksquare ; v ; \blacksquare ; w, \qquad k, \qquad f ; (\blacksquare \otimes \blacksquare) ; g, \qquad p \mid q .$$

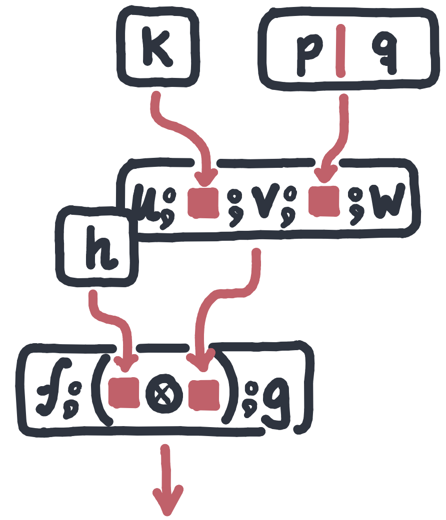
These contexts form a produoidal category.



$$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad N \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad I \to \begin{bmatrix}A\\B\end{bmatrix}$$

MONOIDAL CONTEXT

$$f ⨾ (h \otimes (u ⨾ k ⨾ v ⨾ p ⨾ q ⨾ w)) ⨾ g \quad =$$



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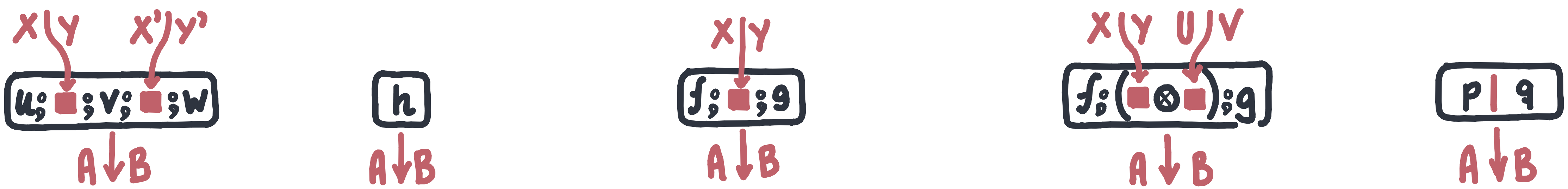


$$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix} \qquad N \to \begin{bmatrix} A \\ B \end{bmatrix} \qquad \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix} \qquad \begin{bmatrix} X \\ Y \end{bmatrix} \otimes \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix} \qquad I \to \begin{bmatrix} A \\ B \end{bmatrix}$$

Monoidal Context-Contour

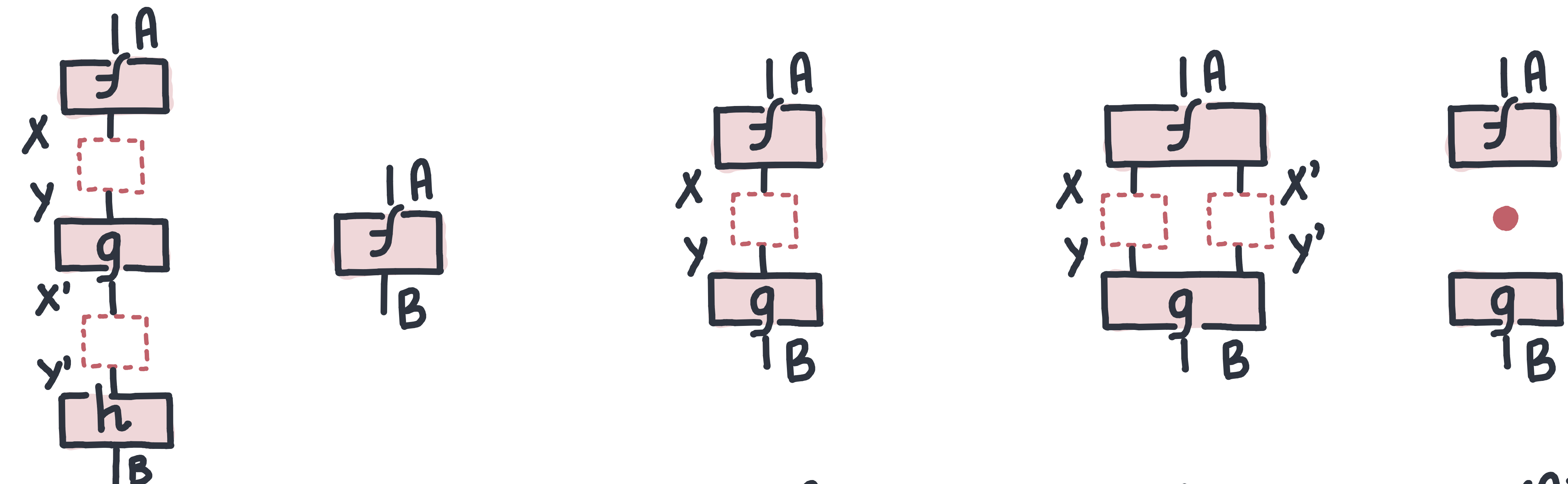
What is a canonical algebra of decomposition on top of a monoidal category?

- Each monoidal category gives a cofree produoidal, monoidal context.
- Each produoidal gives a free monoidal category, monoidal contour.

$$\begin{tikzcd} \text{Monoidal Category} \arrow[r, bend left, "\text{Monoidal Context}"] \arrow[r, phantom, "\top"] & \text{Produoidal Category} \arrow[l, bend left, "\text{Monoidal Contour}"] \end{tikzcd}$$

Monoidal Context

Thm (EHR'23). Spliced string diagrams are the cofree produoidal on a monoidal.

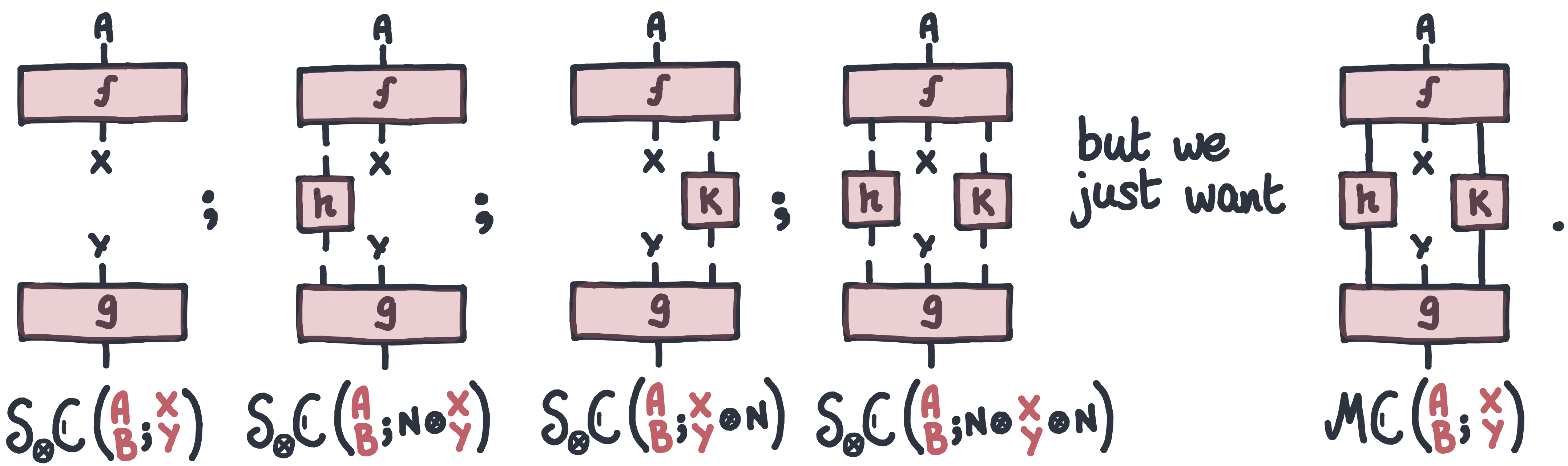


$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad N \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad I \to \begin{bmatrix}A\\B\end{bmatrix}$

Missing

Spliced string diagrams have some issues.

- They separate sequential and parallel units unnecessarily.
- Produoidals introduce a lot of bureaucracy on units.



Part 4: Normalization

Optics for Monoidal Categories

Normalizing Produoidals

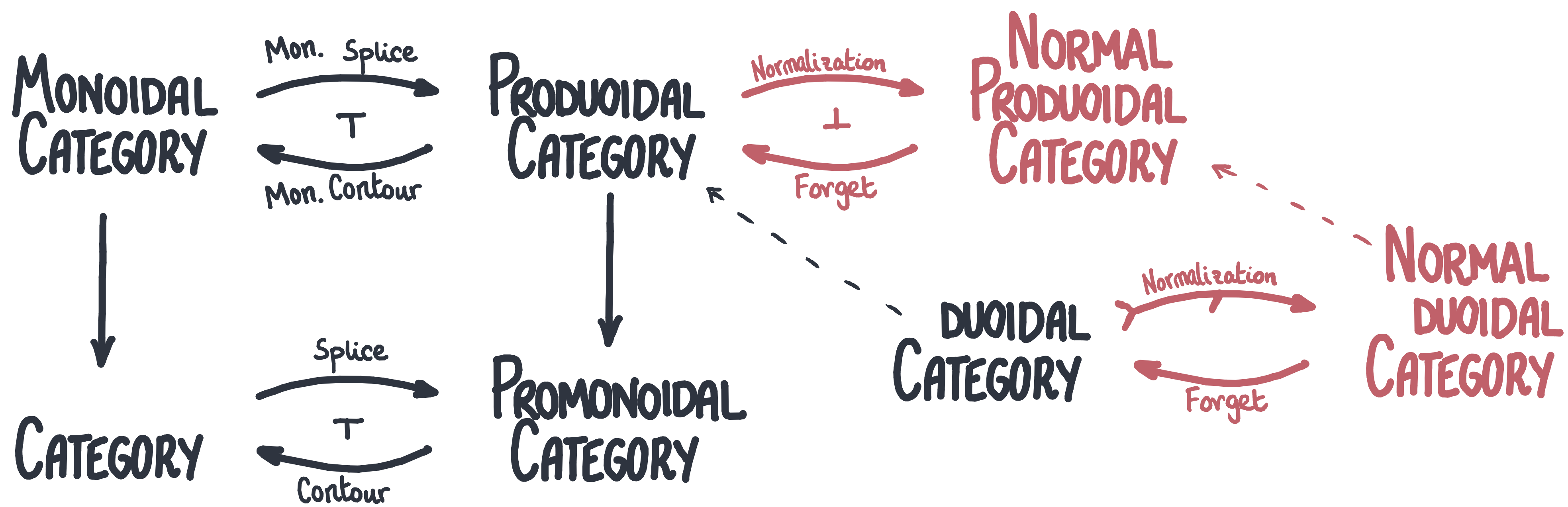
Theorem (EHR23). We can ALWAYS normalize a produoidal category. Moreover, Normalization : Produo → Produo is an idempotent monad, constructing a free normalization. Similarly for symmetric produoidals.

Every duoidal is indeed normalizable, but the result may be a produoidal.

$$\mathcal{N}\mathbb{V}(X;Y) = \mathbb{V}(X; N \otimes Y \otimes N),$$
$$\mathcal{N}\mathbb{V}(X; Y \triangleleft_N Z) = \mathbb{V}(X; (N \otimes Y \otimes N) \triangleleft (N \otimes Z \otimes N)),$$
$$\mathcal{N}\mathbb{V}(X; Y \otimes_N Z) = \mathbb{V}(X; N \otimes Y \otimes N \otimes Z \otimes N),$$
$$\mathcal{N}\mathbb{V}(X; N_N) = \mathcal{N}\mathbb{V}(X; I_N) = \mathbb{V}(X; N),$$

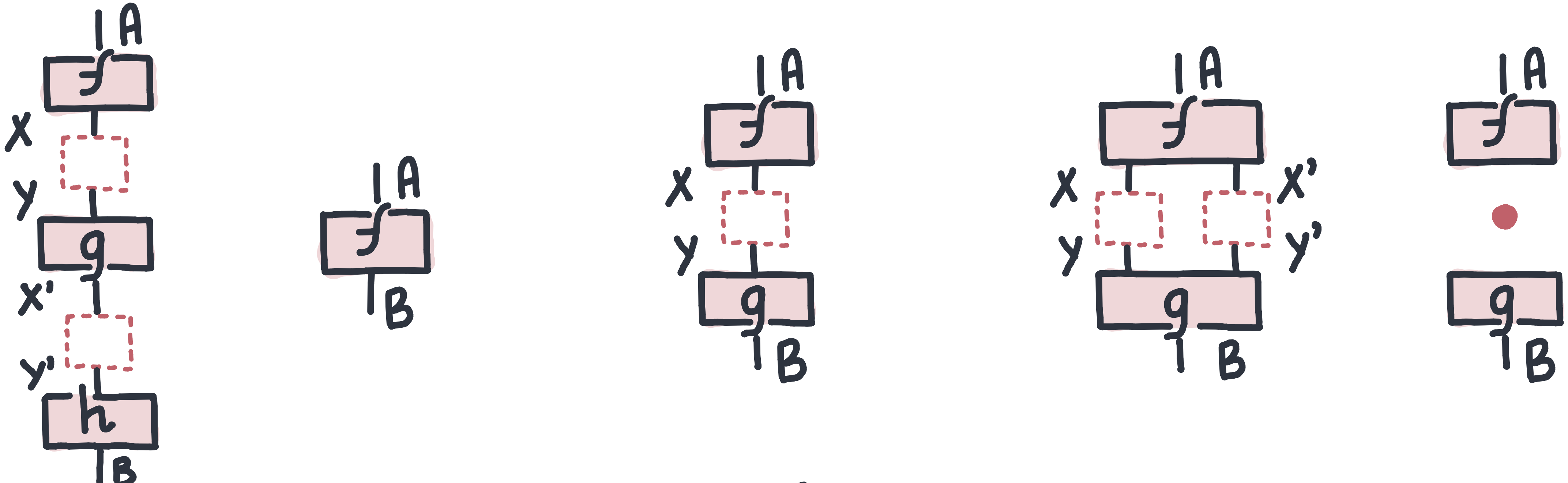
$$\mathcal{N}_\sigma\mathbb{V}(X;Y) = \mathbb{V}(X; N \otimes Y),$$
$$\mathcal{N}_\sigma\mathbb{V}(X; Y \triangleleft_N Z) = \mathbb{V}(X; (N \otimes Y) \triangleleft (N \otimes Z)),$$
$$\mathcal{N}_\sigma\mathbb{V}(X; Y \otimes_N Z) = \mathbb{V}(X; N \otimes Y \otimes Z),$$
$$\mathcal{N}_\sigma\mathbb{V}(X; N_N) = \mathcal{N}_\sigma\mathbb{V}(X; I_N) = \mathbb{V}(X; N).$$

Next


Monoidal Category
Mon. Splice
Mon. Contour
Produoidal Category
Normalization
Forget
Normal Produoidal Category
Category
Splice
Contour
Promonoidal Category
Duoidal Category
Normalization
Forget
Normal Duoidal Category

Monoidal Context

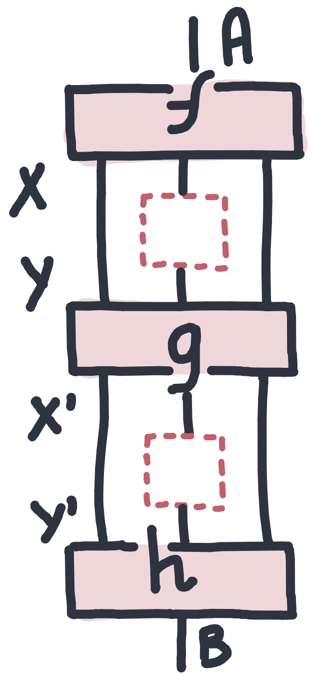
THM (EHR'23). Spliced string diagrams is the cofree produoidal on a monoidal.



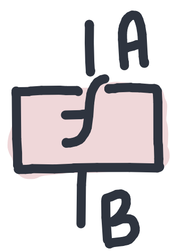
$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad N \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad I \to \begin{bmatrix}A\\B\end{bmatrix}$

Normalized Monoidal Context

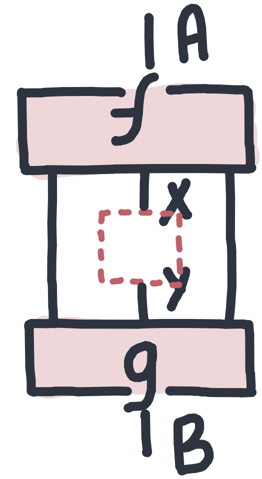
THM (EHR'23). Monoidal optics are the free normalization of the cofree produoidal.



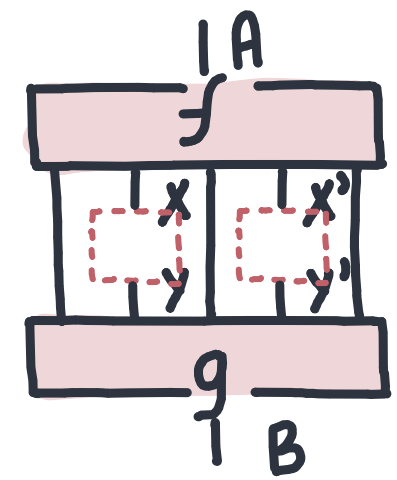
$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$



$\begin{bmatrix}A\\B\end{bmatrix}$

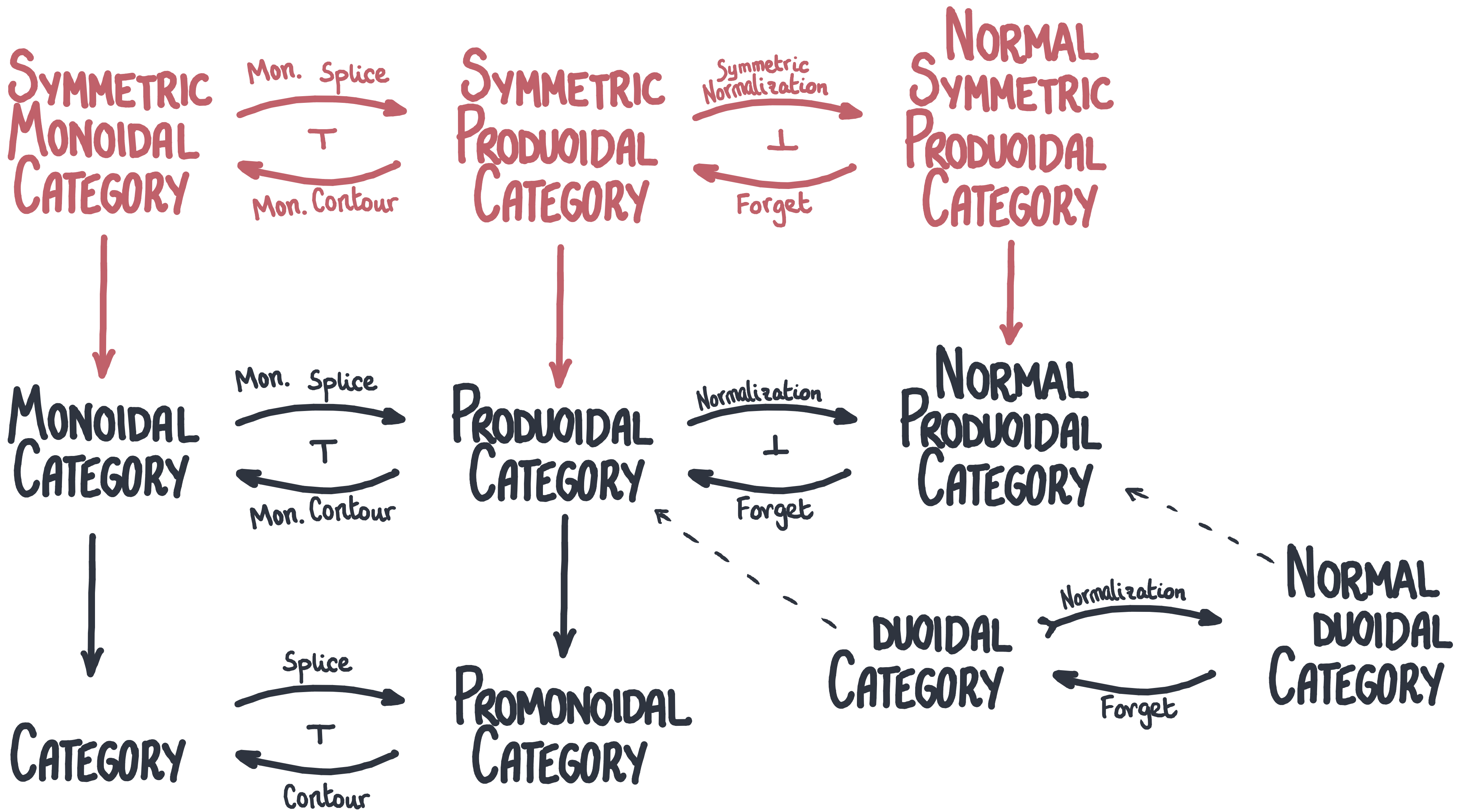


$\begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$



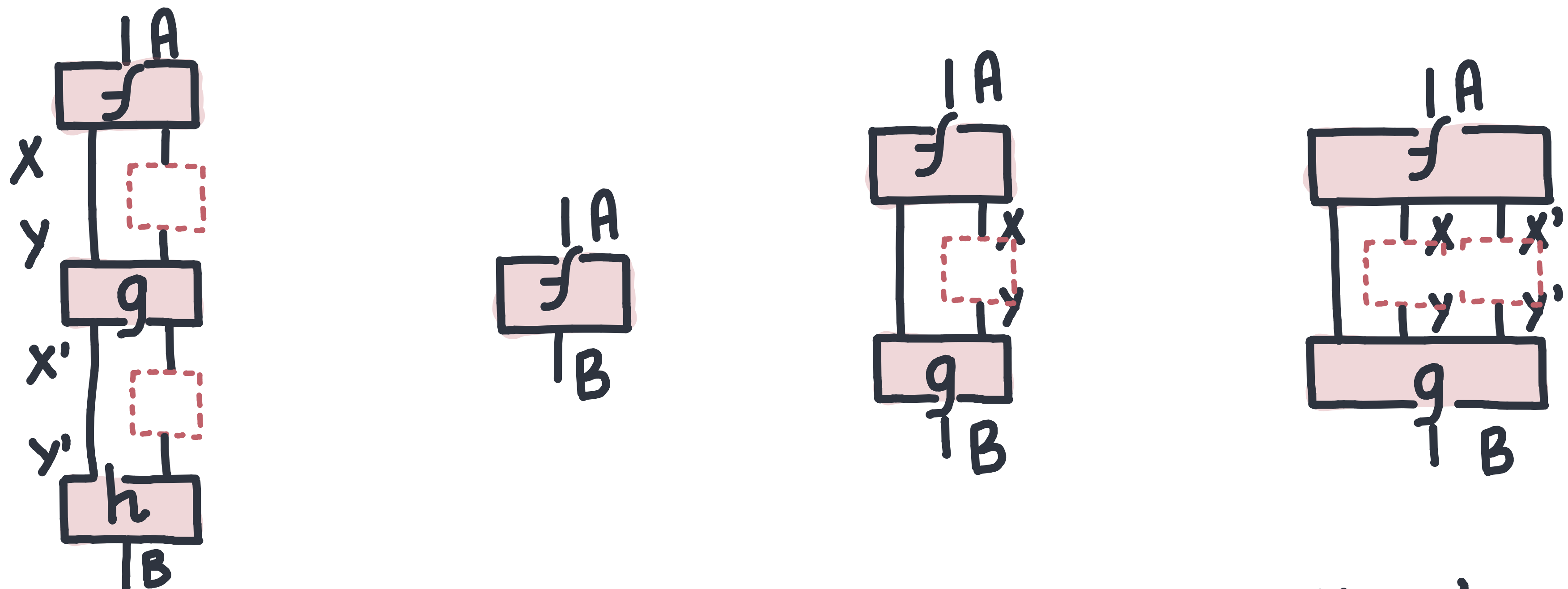
$\begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$

NEXT


SYMMETRIC MONOIDAL CATEGORY
Mon. Splice
⊤
Mon. Contour
SYMMETRIC PRODUOIDAL CATEGORY
Symmetric Normalization
⊥
Forget
NORMAL SYMMETRIC PRODUOIDAL CATEGORY
MONOIDAL CATEGORY
Mon. Splice
⊤
Mon. Contour
PRODUOIDAL CATEGORY
Normalization
⊥
Forget
NORMAL PRODUOIDAL CATEGORY
DUOIDAL CATEGORY
Normalization
Forget
NORMAL DUOIDAL CATEGORY
CATEGORY
Splice
⊤
Contour
PROMONOIDAL CATEGORY

Normalized Symmetric Monoidal Context

Thm (EHR'23). Optics are the free sym. normalization of the cofree produoidal.

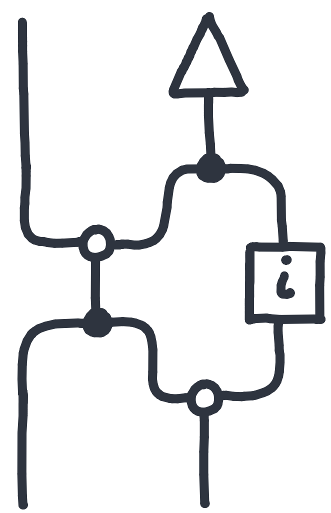


$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad$ $\begin{bmatrix}A\\B\end{bmatrix}$ $\quad$ $\begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad$ $\begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$

Part 5: Example

ONE-TIME PAD

Broadbent & Karvonen propose a formalization of the one-time pad in a monoidal category with a Hopf algebra with an integral.



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Broadbent & Karvonen. Categorical Composable Cryptography.

ONE-TIME PAD

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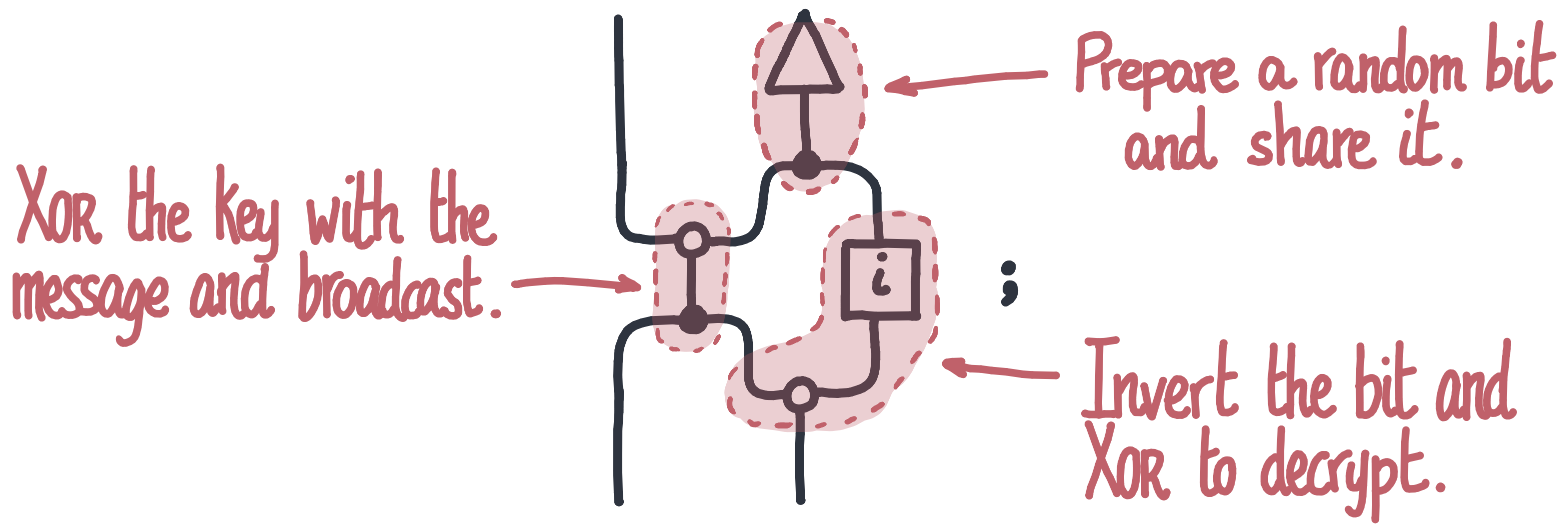

Broadbent & Karvonen. Categorical Composable Cryptography.

ONE-TIME PAD

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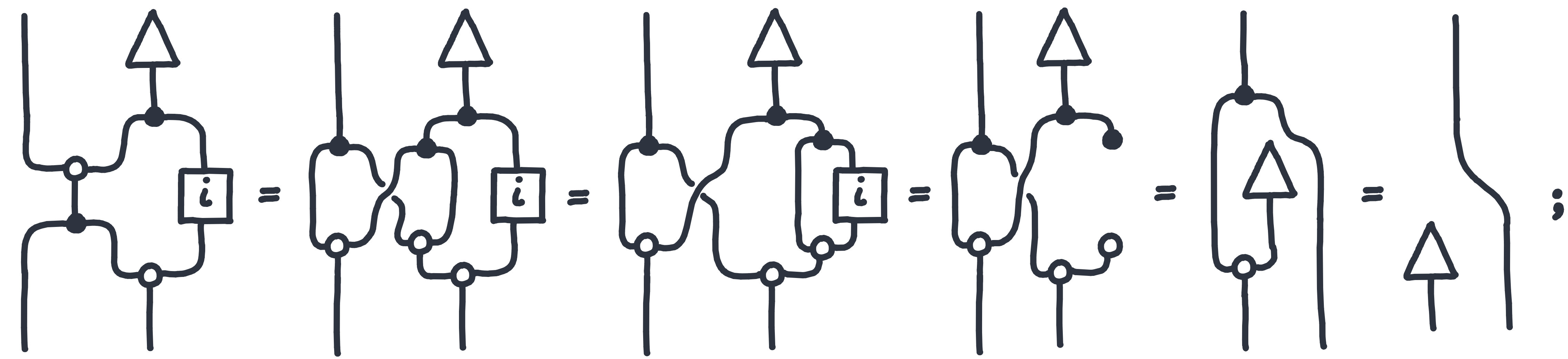
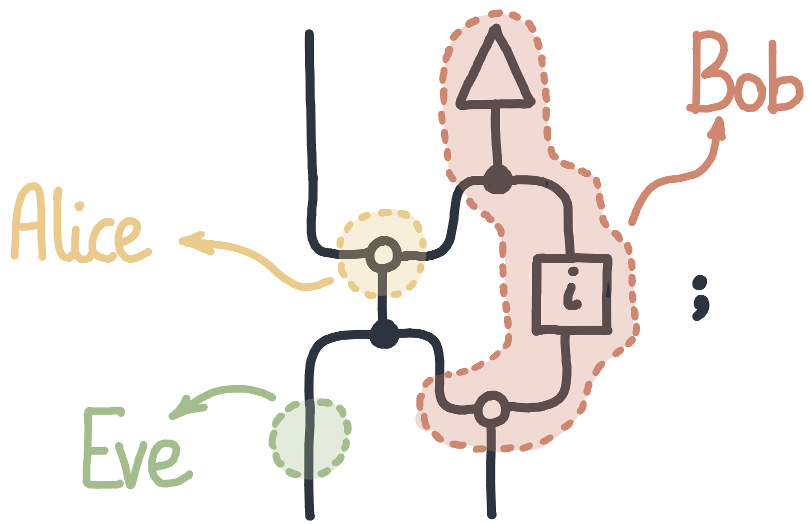


We can reason about security using string diagrams.

Broadbent & Karvonen. Categorical Composable Cryptography.

ONE-TIME PAD

We want to split the morphism into different agents: Alice does not control the broadcast; Eve can only attack at the end; Bob keeps a bit in memory.



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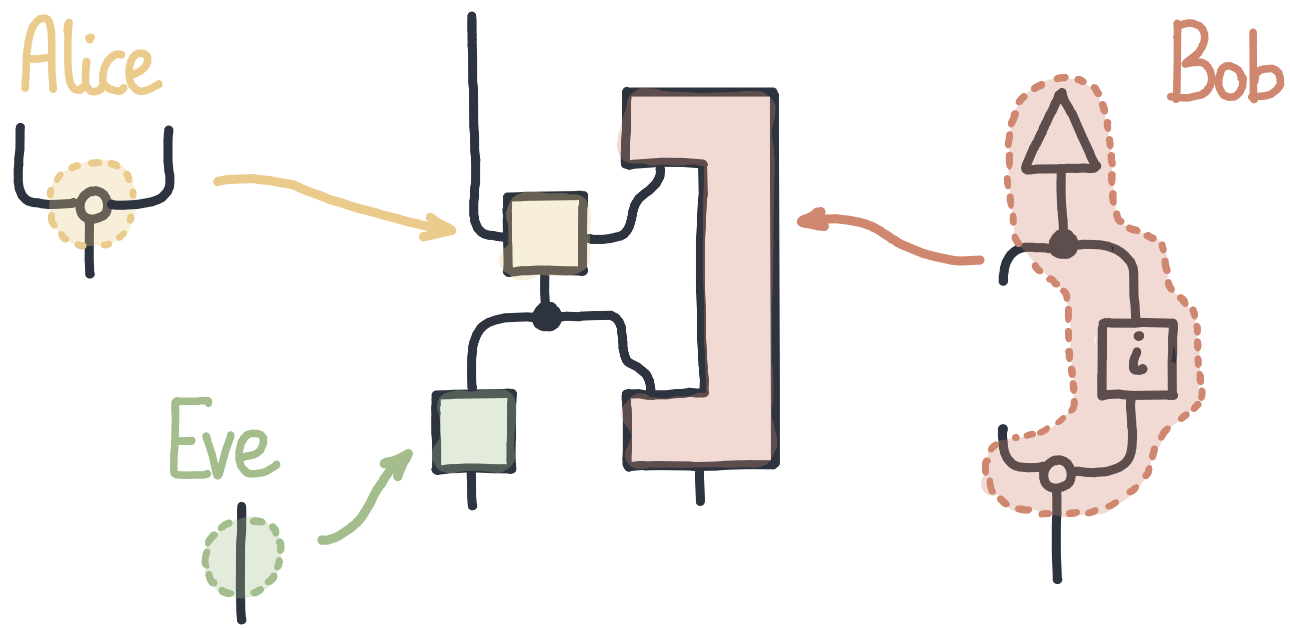
ONE-TIME PAD

We want to split the morphism into different agents: Alice does not control the broadcast; Eve can only attack at the end; Bob keeps a bit in memory.



The set of possible actions of Alice and Eve are given by a hom-set; they are monoidal morphisms. What about Bob?

One-Time Pad

This is not only about string diagrams; this is about code modularity and separation.

```
oneTimePad(msg) = do
  key <- randomBit
  crypt <- xor(msg,key)
  msg <- xor(crypt,key)
  return msg
```

Do-notation is a syntax for (pre)monoidal categories; following string diagrams.
We can extend it with message-passing, and split into components.

Heunen & Jacobs, Hughes, Staton & Levy, Román.

ONE-TIME PAD

github.com/mroman42/one-time-pad-example

This is not only about string diagrams; this is about code modularity and separation.

```
oneTimePad(alice,bob,eve,msg) = do
  key <- bob₀()
  crypt <- alice(msg, key)
  () <- eve(crypt)
  msg <- bob₁(crypt)
  return msg
```

```
alice(msg, key) = do
  crypt <- xor(msg,key)
  return crypt
```

```
bob() = do
  key <- randomBit
  !key
  ?crypt
  msg <- xor(crypt,key)
  return msg
```

```
eve(crypt) = do
  return crypt
```

FURTHER WORK

- Monoidal Context Theory. Mario Román, PHD Thesis.
- Deadlock-free Message Passing via Polar Shufflings. Matt Earnshaw, Chad Nester, Mario Román; extended abstract for NWPT'23.
- A Monoidal Chomsky-Schützenberger Theorem. Matt Earnshaw, Mario Román; in peer-review.

End

NORMALIZING DUOIDALS

A duoidal $(\triangleleft, N, \otimes, I)$ is **normal** whenever $I \xrightarrow{\cong} N$.

- Being normal is a property (idempotent monad?).
- However, we cannot normalize any duoidal.

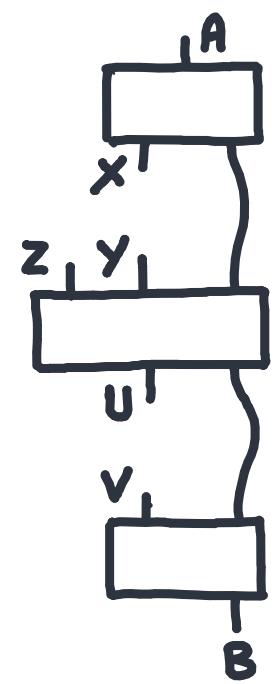
THEOREM (Garner, López Franco). Let $(V, \otimes, I, \triangleleft, N)$ a duoidal with reflexive coequalizers, preserved by $(\otimes)$. Then, $(\mathrm{Bimod}^{\otimes}_{N}, \otimes_N, N, \triangleleft, N)$ is a normal duoidal. Similarly for symmetric duoidals.

Garner & López Franco. Commutativity.

FURTHER WORK

In the category of lenses, we can write exchanges, e.g.

$$\mathcal{LC}\left(\begin{matrix}A\\B\end{matrix};\left(\begin{matrix}X\\Y\end{matrix}\otimes\begin{matrix}I\\Z\end{matrix}\right)\triangleleft\begin{matrix}U\\V\end{matrix}\right)$$



PROPOSITION. The $\otimes$ of lenses is representable. Lenses are monoidal with $\left(\begin{matrix}X\\Y\end{matrix}\right)\otimes\left(\begin{matrix}X'\\Y'\end{matrix}\right)=\left(\begin{matrix}X\otimes X'\\Y\otimes Y'\end{matrix}\right)$.

$$\mathcal{LC}\left(\begin{matrix}A\\B\end{matrix};\ \begin{matrix}X\\Y\otimes Z\end{matrix}\triangleleft\begin{matrix}U\\V\end{matrix}\right)$$

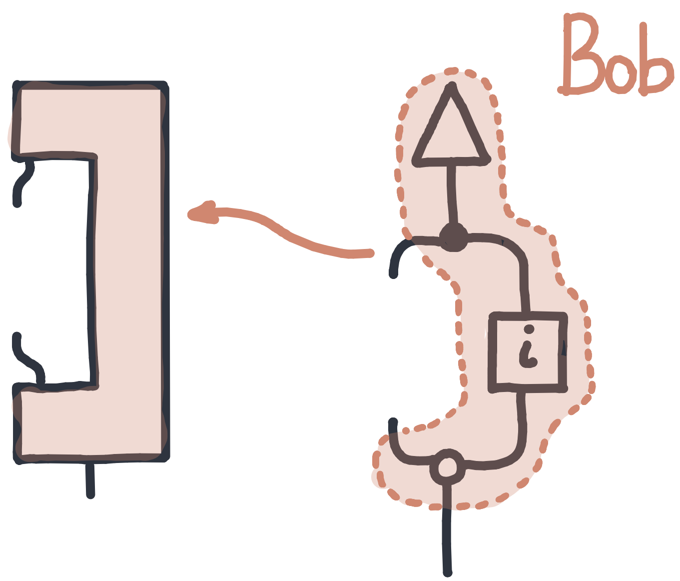
PROPOSITION. There exist mon. functors $(!):\mathcal{C}\to\mathcal{LC}$ and $(?):\mathcal{C}^{op}\to\mathcal{LC}$. These satisfy $!X=\left(\begin{matrix}X\\I\end{matrix}\right)$, $?X=\left(\begin{matrix}I\\X\end{matrix}\right)$, with $\left(\begin{matrix}X\\Y\end{matrix}\right)=!X\otimes ?Y=!X\triangleleft ?Y$,

$$\mathcal{LC}\left(\begin{matrix}A\\B\end{matrix};\ !X\triangleleft ?(Y\otimes Z)\triangleleft !U\triangleleft ?V\right).$$

! SEND
? RECEIVE

Honda et al.

SUMMARY



What are these incomplete diagrams?

M.R. Open Diagrams via Coend Calculus.

Which structure do they form?

Matt Earnshaw, James Hefford, M.R.
The Produoidal Algebra of Process Decomposition.

Summary

- Part 1. Profunctors
- Part 2. Promonoidal Categories
- Part 3. Context for Categories
- Part 4. Context for Monoidal Categories
- Part 5. Normal Context for Monoidal Categories
- Part 6. Normal Context for Symmetric Monoidal Categories
- Part 7. Send/Receive Session Types

Monoidal Category

DEFINITION. A monoidal category is a category $\mathbb{C}$ together with functors

$$(\otimes) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \qquad I : \mathbb{1} \to \mathbb{C},$$

and natural isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$
$$\lambda_A : I \otimes A \to A,$$
$$\rho_A : A \otimes I \to A,$$

satisfying the pentagon and triangle equations.

By nesting, $X \otimes (Y \otimes Z)$, we mean functor composition,

$$X \otimes (Y \otimes Z) :=$$
$$X \otimes M \text{ where } M = Y \otimes Z.$$

PROMONOIDAL CATEGORY

DEFINITION. A promonoidal category is a category $\mathbb{C}$ together with profunctors

$$\mathbb{C}(\bullet \otimes \bullet ; \bullet) \colon \mathbb{C}^{op} \times \mathbb{C} \times \mathbb{C} \to \mathrm{SET}, \qquad \mathbb{C}(I ; \bullet) \colon \mathbb{C}^{op} \to \mathrm{SET},$$

and natural bijections,

$$\alpha_{A,B,C} : \mathbb{C}(X \otimes (Y \otimes Z) ; \bullet) \to \mathbb{C}((X \otimes Y) \otimes Z ; \bullet),$$
$$\lambda_A : \mathbb{C}(I \otimes X ; \bullet) \to \mathbb{C}(X ; \bullet),$$
$$\rho_A : \mathbb{C}(X \otimes I ; \bullet) \to \mathbb{C}(X ; \bullet),$$

satisfying the pentagon and triangle equations.

By nesting, $\mathbb{C}(X \otimes (Y \otimes Z) ; \bullet)$, we mean profunctor composition,

$$\mathbb{C}(X \otimes (Y \otimes Z) ; \bullet) := \int^{M} \mathbb{C}(X \otimes M ; \bullet) \times \mathbb{C}(Y \otimes Z ; M).$$

Monoidal Context-Contour

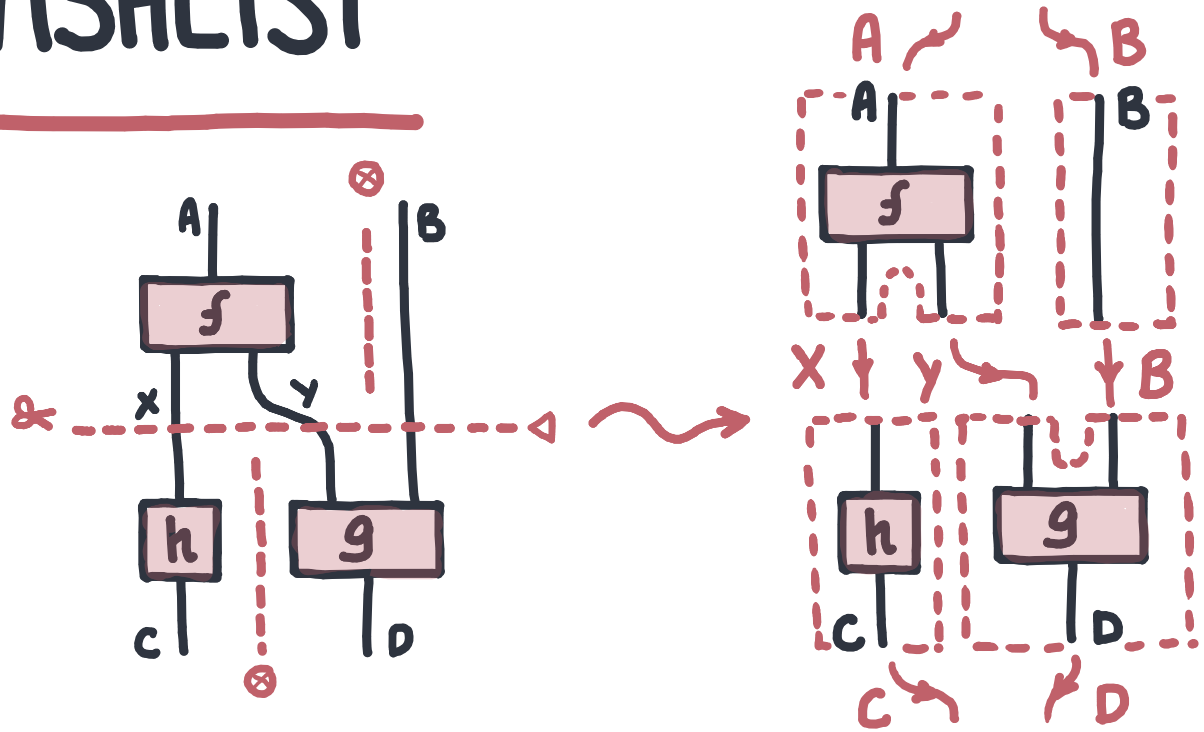
What is a canonical algebra of decomposition on top of a monoidal category?

- Each monoidal category gives a cofree produoidal, monoidal context.
- Each produoidal gives a free monoidal category, monoidal contour.

$$
\begin{tikzcd}
\text{Monoidal Category} \arrow[r, bend left, "\text{Monoidal Context}"] \arrow[r, phantom, "\top"] & \text{Produoidal Category} \arrow[l, bend left, "\text{Monoidal Contour}"]
\end{tikzcd}
$$

PART 1 : PROFUNCTORS, DINATURALITY

WISHLIST



$A \otimes B$ splits into
$(A \otimes B) \triangleleft ((X \otimes Y) \otimes B) \triangleleft (X \otimes (Y \otimes B)) \triangleleft (C \otimes D)$
to get to $C \otimes D$

Algebra of parallel/sequential decomposition. (Pro)duoidal categories.

Semantics of incomplete diagrams. Monoidal context.

Message-passing. Send-Receive types.

PROFUNCTORS (FUNCTORS $\mathbb{C}^{op} \times \mathbb{D} \to$ SET)

Profunctors are sets of processes $P(X;Y)$ indexed contravariantly by a inputs $X \in \mathbb{C}^{op}$ and covariantly by outputs, $Y \in \mathbb{D}$. They have actions,

$$(>) : \mathbb{C}(X';X) \times P(X;Y) \to P(X';Y),$$
$$(<) : P(X;Y) \times \mathbb{D}(Y;Y') \to P(X;Y'),$$

satisfying

$$f > (p < g) = (f > p) < g \; ; \qquad \text{compatibility}$$
$$f_0 > f_1 > p = (f_0 \fatsemi f_1) > p \; ; \quad id > p = p \; ; \qquad \text{left action}$$
$$p < g_0 < g_1 = p < (g_0 \fatsemi g_1) \; ; \quad p < id = p. \qquad \text{right action}$$

Normalizing Symmetric Duoidals

A duoidal $(\triangleleft, N, \otimes, I)$ is $\otimes$-symmetric whenever $\otimes, I$ is symm. monoidal.

- We can normalize symmetric duoidals as usual.
- However, there is a more specialized procedure.

THEOREM (Garner, López Franco). Let $(V, \otimes, I, \triangleleft, N)$ a sym$^{\otimes}$. duoidal with reflexive coequalizers, preserved by $(\otimes)$. Then, $(\mathrm{Mod}^{\otimes}_{N}, \otimes_N, N, \triangleleft, N)$ is a normal sym$^{\otimes}$. duoidal.

Garner & López Franco. Commutativity.

Profunctor Composition

A process of the composite type $P ; Q$ is a process in P communicating with one in Q. That is, $\langle p \mid q \rangle \in P ; Q(X, Z)$ is given by $p \in P(X, Y)$ followed by $q \in Q(Y, Z)$ for some Y.

Given $p \in P(X; Y_0)$, $q \in Q(Y_1; Z)$, and a morphism $f \in \mathbb{B}(Y_0; Y_1)$, we can

- execute p to get a Y_0, then execute q with f,
- execute p with f to get a Y_1, then execute q.

$$\langle p < f \mid q \rangle \sim_d \langle p \mid f > q \rangle.$$

These are "dinaturally" equivalent.

Coends are the colimits that arise by quotienting by dinatural equivalence.

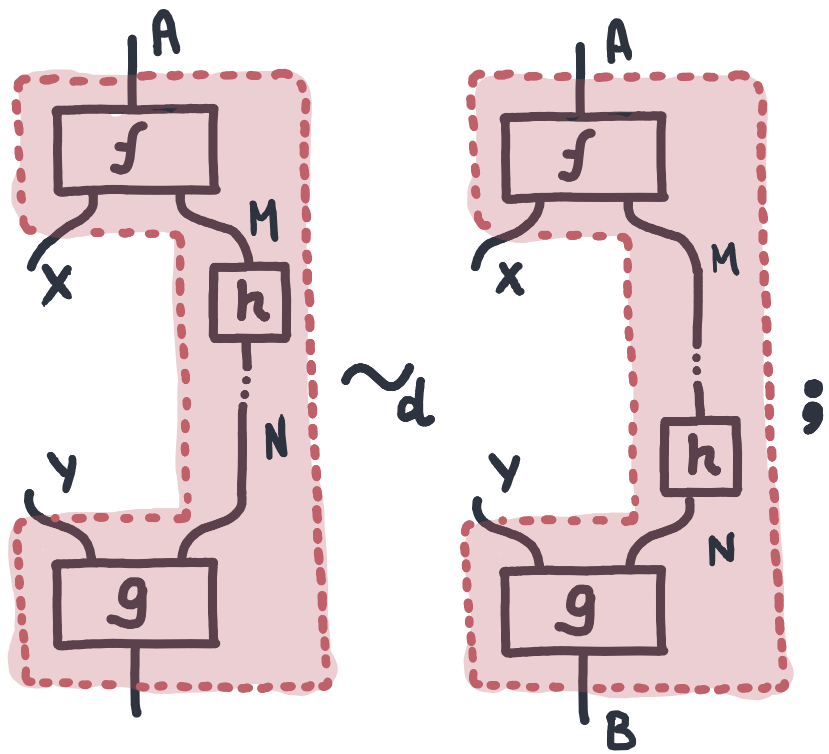
$$\int^{Y \in \mathbb{B}} P(X; Y) \times Q(Y; Z)$$

$$:=$$

$$\coprod_{Y \in \mathbb{B}} P(X; Y) \times Q(Y; Z) / \sim_d .$$

DINATURALITY

We could define contexts as pairs of morphisms, but we would like the following two to be equal.



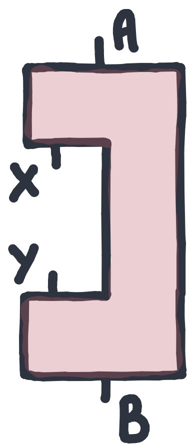
DEFINITION. A context of type $(A;B)$ with a hole $(X;Y)$ is a pair of morphisms $\langle f,g\rangle_M$ with
$f : A \to M \otimes X$ and $g : M \otimes Y \to B$,
quotiented by dinaturality.

$$\langle f \mathbin{;} (h \otimes id) \mid g \rangle \sim_d \langle f \mid (h \otimes id) \mathbin{;} g \rangle.$$

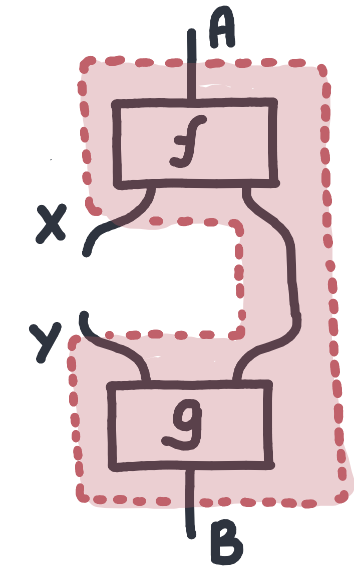
In other words, it is an element of

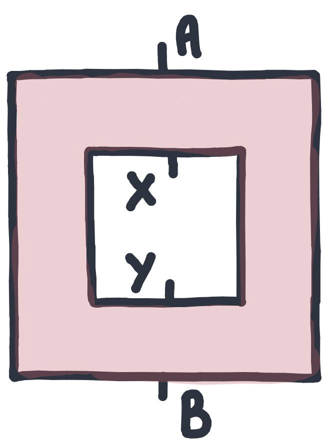
$$\int^{M \in \mathbb{C}} \mathbb{C}(A; X \otimes M) \times \mathbb{C}(Y \otimes M; B).$$

Open Diagrams

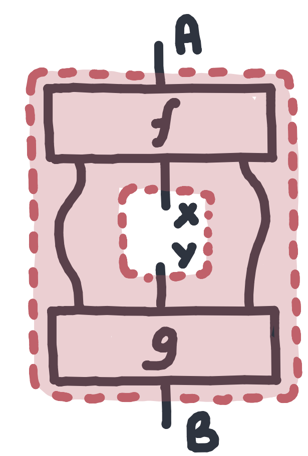


$$\rightsquigarrow \int^{M} \hom(A; X \otimes M) \times \hom(Y \otimes M; B) \rightsquigarrow$$

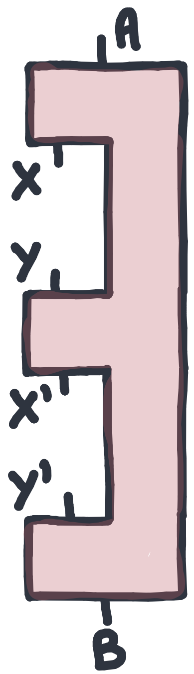




$$\rightsquigarrow \int^{M} \hom(A; M \otimes X \otimes N) \times \hom(M \otimes Y \otimes N; B) \rightsquigarrow$$

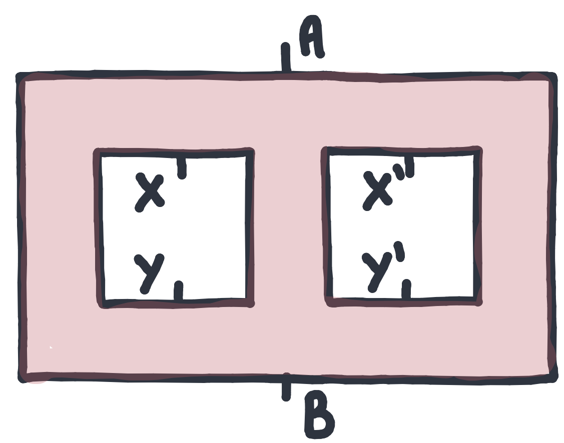


Open Diagrams

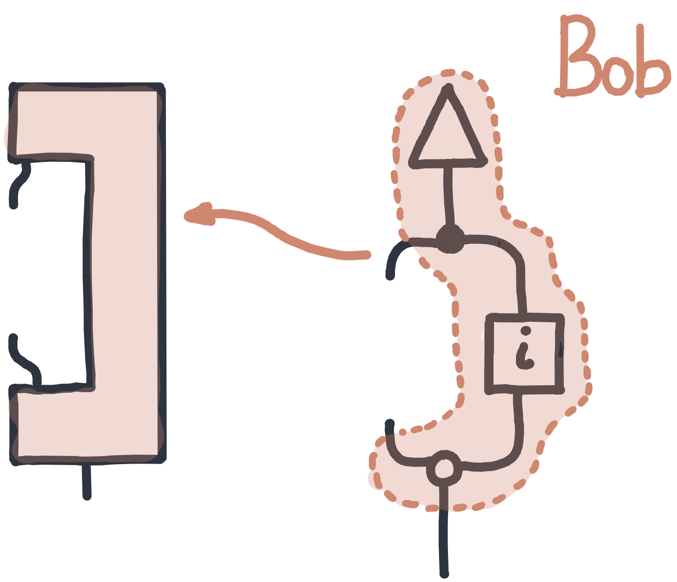


$$\int^{M} \mathrm{hom}(A; X \otimes M) \times \mathrm{hom}(Y \otimes M; X' \otimes N) \times \mathrm{hom}(Y' \otimes N; B)$$

$$\int^{M,N,O} \mathrm{hom}(A; M \otimes X \otimes N \otimes X' \otimes O) \times \mathrm{hom}(M \otimes Y \otimes N \otimes Y' \otimes O; B)$$



SUMMARY



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M.R. Open Diagrams via Coend Calculus.

Which structure do they form?

Matt Earnshaw, James Hefford, M.R.
The Produoidal Algebra of Process Decomposition.

Splice-Contour

What is a canonical algebra of decomposition on top of a category?

- Each promonoidal gives a free category. Contour
- Each category gives a cofree promonoidal. Splice

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend right, "\text{Splice}"'] \arrow[r, phantom, "\perp"] & \mathrm{Promon} \arrow[l, bend right, "\text{Contour}"'] \end{tikzcd}$$

Melliès & Zeilberger. Parsing as a Lifting Problem.

Splice-Contour

We can rewrite Melliès & Zeilberger for promonoidals instead of multicategories.

- Universal context: cofree promonoidal.
- Morphisms with holes: $u \mathbin{;} \blacksquare \mathbin{;} v \mathbin{;} \blacksquare \mathbin{;} w$.
- $\mathrm{Splice}(\mathbb{C})$ is the monoid of the duality $\mathbb{C} \dashv \mathbb{C}^{op}$.
- Can we do the same for monoidal categories?

Melliès & Zeilberger. Parsing as a Lifting Problem.

DUOIDALS

An extra dimension side-steps Eckmann-Hilton.

DEFINITION. A duoidal category is a category $\mathbb{V}$ with two promonoidal structures

$\triangleleft : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, "seq. split" $N : 1 \to \mathbb{V}$, "seq unit"

$\otimes : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, "par. split" $I : 1 \to \mathbb{V}$, "par. unit"

such that one laxly distributes over the other,

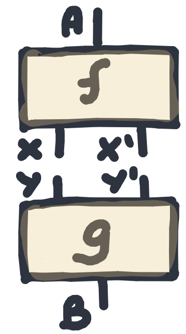
$$\psi_2 : (A \triangleleft B) \otimes (C \triangleleft D) \longrightarrow (A \otimes C) \triangleleft (B \otimes D),$$
$$\psi_0 : I \longrightarrow N,$$
$$\varphi_2 : N \longrightarrow N \triangleleft N,$$
$$\varphi_0 : I \longrightarrow I \otimes I.$$

We ask coherence for these maps.

RECAP: SPLICED MONOIDAL ARROWS

$S_{\otimes}\mathbb{C}\left(\begin{smallmatrix}A\\B\end{smallmatrix};\begin{smallmatrix}X\\Y\end{smallmatrix}\right)$
morphism

$S_{\otimes}\mathbb{C}\left(\begin{smallmatrix}A\\B\end{smallmatrix};\begin{smallmatrix}X\\Y\end{smallmatrix}\triangleleft\begin{smallmatrix}X'\\Y'\end{smallmatrix}\right)$
sequential split



$S_{\otimes}\mathbb{C}\left(\begin{smallmatrix}A\\B\end{smallmatrix};\begin{smallmatrix}X\\Y\end{smallmatrix}\otimes\begin{smallmatrix}X'\\Y'\end{smallmatrix}\right)$
parallel split

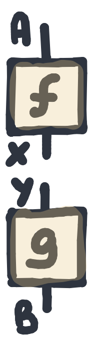
$S_{\otimes}\mathbb{C}\left(\begin{smallmatrix}A\\B\end{smallmatrix};N\right)$
sequential unit

$S_{\otimes}\mathbb{C}\left(\begin{smallmatrix}A\\B\end{smallmatrix};I\right)$
parallel unit

THM (EHR'23). Spliced monoidal arrows are the cofree produoidal on a monoidal.

PART 6 : NORMAL CONTEXT FOR SYMM. MONOIDAL CATEGORIES

Recap: Spliced Monoidal Arrows

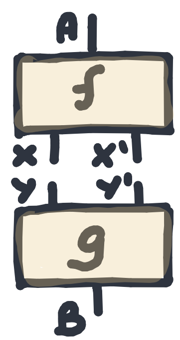


$S_\otimes \mathbb{C}\left(\begin{matrix}A\\B\end{matrix}; \begin{matrix}X\\Y\end{matrix}\right)$

morphism

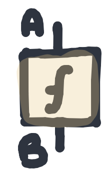
$S_\otimes \mathbb{C}\left(\begin{matrix}A\\B\end{matrix}; \begin{matrix}X\\Y\end{matrix} \triangleleft \begin{matrix}X'\\Y'\end{matrix}\right)$

sequential split



$S_\otimes \mathbb{C}\left(\begin{matrix}A\\B\end{matrix}; \begin{matrix}X\\Y\end{matrix} \otimes \begin{matrix}X'\\Y'\end{matrix}\right)$

parallel split



$S_\otimes \mathbb{C}\left(\begin{matrix}A\\B\end{matrix}; N\right)$

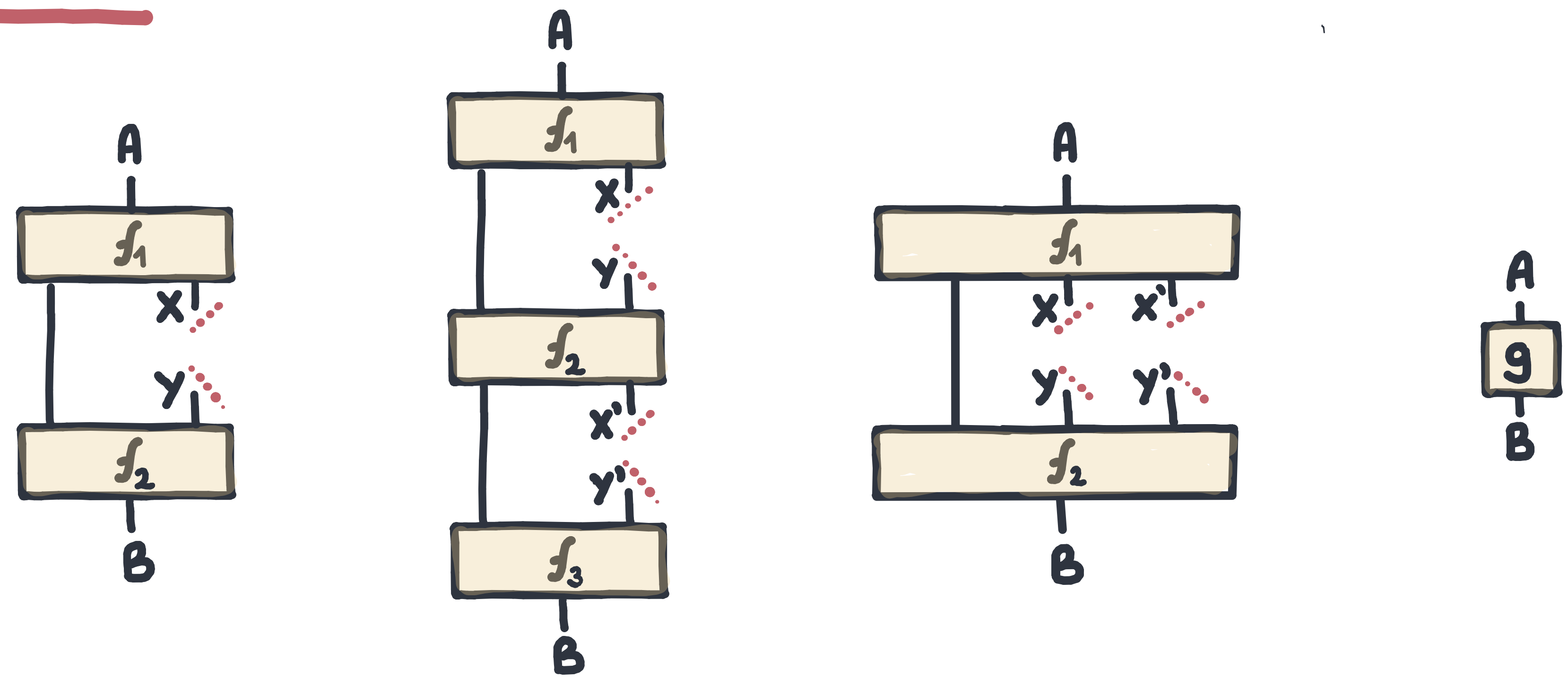
sequential unit

$S_\otimes \mathbb{C}\left(\begin{matrix}A\\B\end{matrix}; I\right)$

parallel unit

THM (EHR'23). Spliced monoidal arrows are the cofree produoidal on a monoidal.

Monoidal Lenses



$\mathcal{LC}\left(\begin{smallmatrix}A\\B\end{smallmatrix}; \begin{smallmatrix}X\\Y\end{smallmatrix}\right)$
morphism

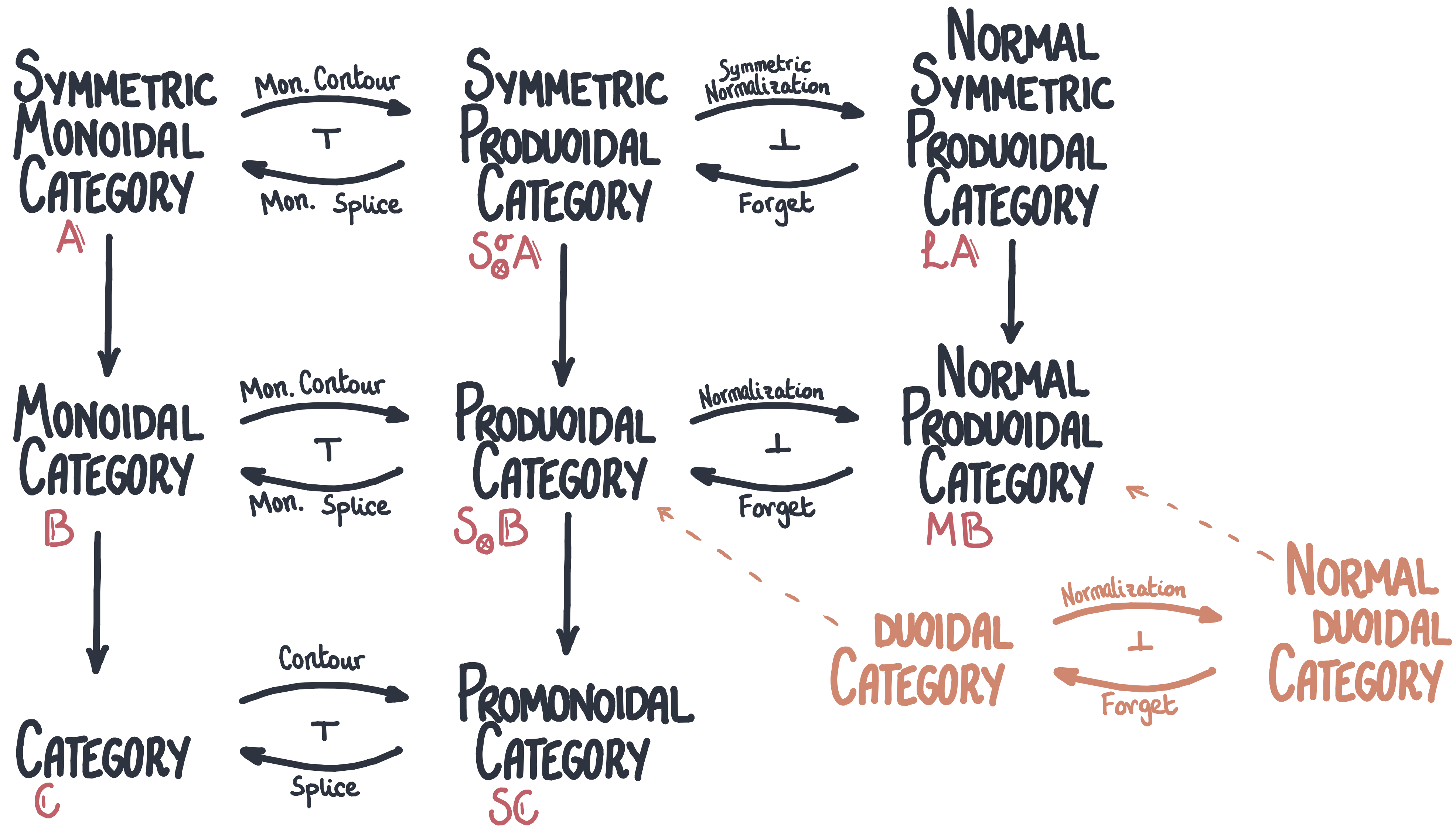
$\mathcal{LC}\left(\begin{smallmatrix}A\\B\end{smallmatrix}; \begin{smallmatrix}X\\Y\end{smallmatrix} \triangleleft \begin{smallmatrix}X'\\Y'\end{smallmatrix}\right)$
sequential split

$\mathcal{LC}\left(\begin{smallmatrix}A\\B\end{smallmatrix}; \begin{smallmatrix}X\\Y\end{smallmatrix} \otimes \begin{smallmatrix}X'\\Y'\end{smallmatrix}\right)$
parallel split

$\mathcal{LC}\left(\begin{smallmatrix}A\\B\end{smallmatrix}; N\right)$
sequential unit

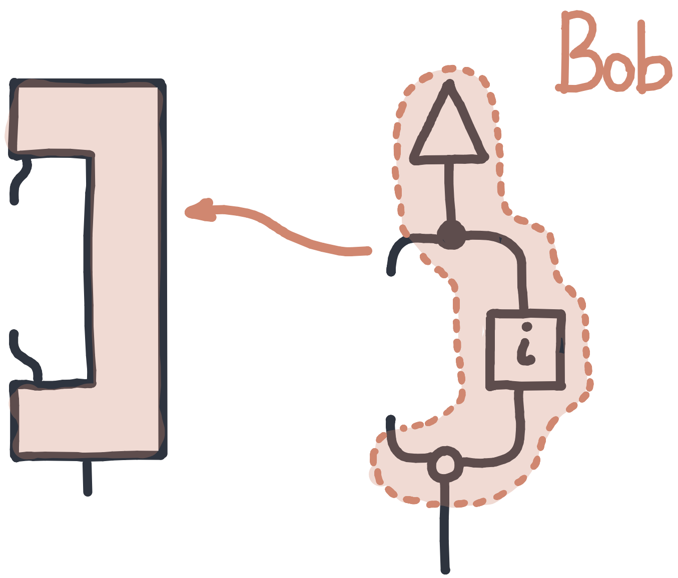
THM (EHR'23). Spliced monoidal arrows are the cofree produoidal on a monoidal.

Recap


Symmetric Monoidal Category
$\mathbb{A}$
Mon. Contour
⊤
Mon. Splice
Symmetric Produoidal Category
$S^{\sigma}_{\otimes}\mathbb{A}$
Symmetric Normalization
⊥
Forget
Normal Symmetric Produoidal Category
$\mathcal{L}\mathbb{A}$
Monoidal Category
$\mathbb{B}$
Mon. Contour
⊤
Mon. Splice
Produoidal Category
$S_{\otimes}\mathbb{B}$
Normalization
⊥
Forget
Normal Produoidal Category
$\mathcal{M}\mathbb{B}$
Category
$\mathbb{C}$
Contour
⊤
Splice
Promonoidal Category
$S\mathbb{C}$
Duoidal Category
Normalization
⊥
Forget
Normal Duoidal Category

Part 7 : Session types

One-Time Pad



$\mathrm{Bob} : \mathrm{LDist}(I \to B \,;\, !B \triangleleft ?B)$

```
bob() = do
  key <- randomBit
  !key
  ?crypt
  msg <- xor(crypt,key)
  return msg
```

We can finally type pieces of a morphism and compose them via the laxators of a produoidal category, or simply by string diagrams.

Some References

- Garner, López Franco. Commutativity.
- Melliès, Zeilberger. Parsing as a lifting problem.
- Pastro, Street. Doubles for monoidal categories.
- Booker, Street. Tannaka duality and convolution for duoidal categories.

- Earnshaw, Hefford, Román. The Produoidal Algebra of Process Decomposition.
- Román. Open Diagrams via Coend Calculus.

Relation to: Wiring Diagrams

Spivak, Vasilakopoulou, Rupel, ...

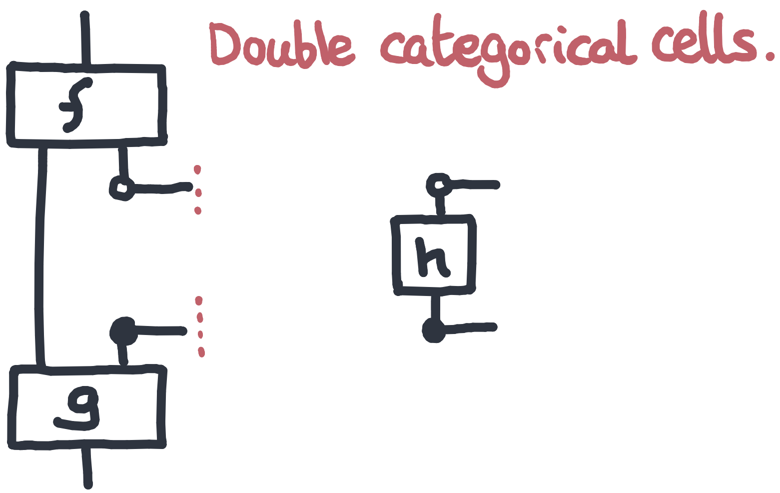
There is a procedure that, from a normal symmetric produoidal extracts a symmetric multicategory: it forgets dependency.

$$\begin{tikzcd}
 & \mathbb{V}(A;B \triangleleft C) \arrow[dr] \arrow[dd, phantom, "<"] & \\
\mathbb{V}(A;B \otimes C) \arrow[ur] \arrow[dr] & & \mathbb{V}(A;B \boxtimes C) \\
 & \mathbb{V}(A;C \triangleleft B) \arrow[ur] &
\end{tikzcd}$$

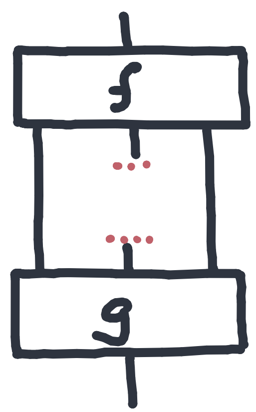
- The multicategory of wiring diagrams is the normal symmetric produoidal of lenses, after forgetting dependency.

Relation to: Cornering Diagrams

We can also split processes in the free proarrow equipment.



Limited to the symmetric case. The following is not expressible.



THM (Nester, Boisseau, Román). In the symmetric case, contexts and one-sided cornering cells coincide. Thus, they form the free normal symm. cofree produoidal.

RELATION TO: LINEARLY DISTRIB. CATS.

Linear actegories provide semantics for concurrency. Can we compare?

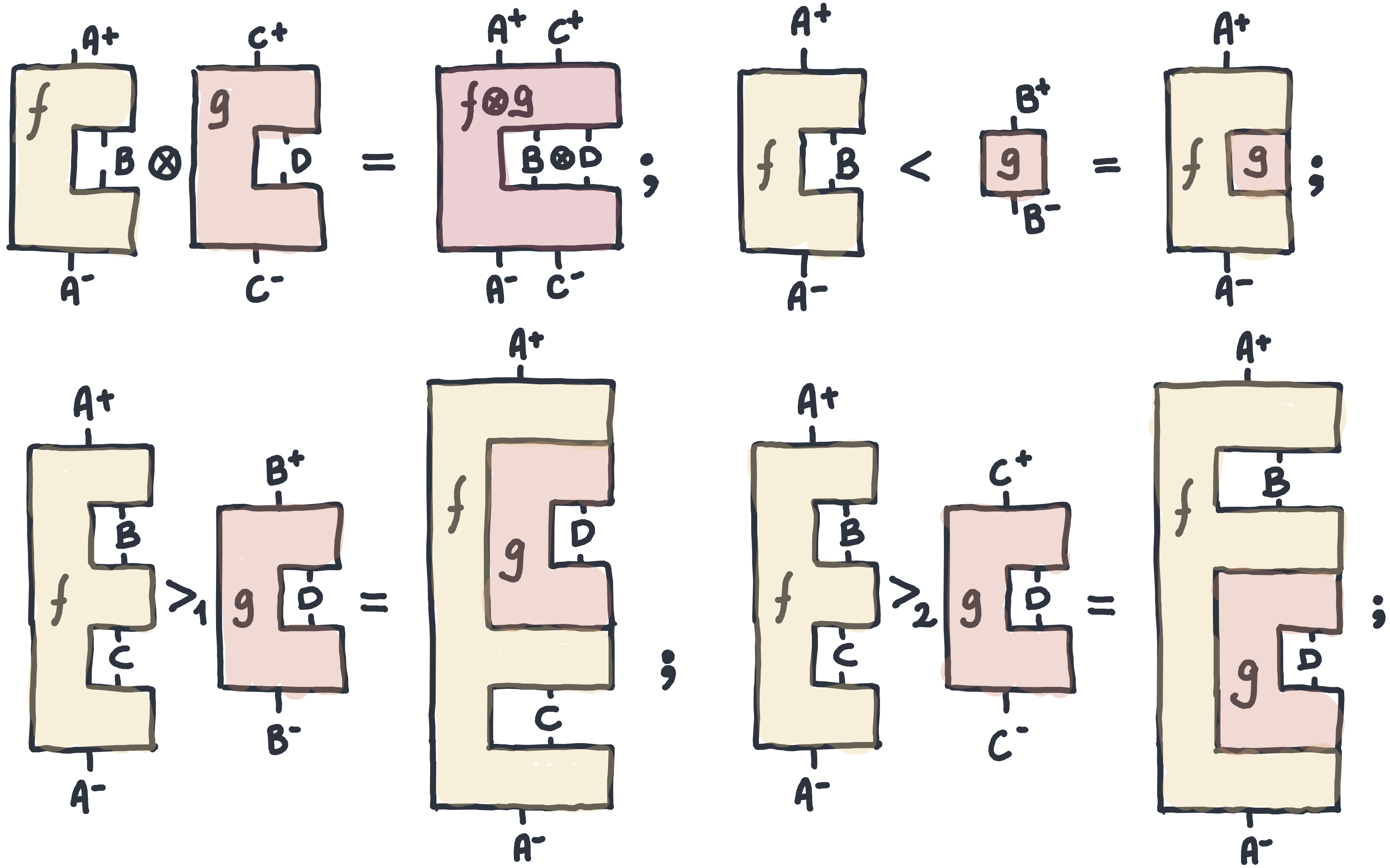
- Types track polarities assigned by the user, instead of send/receive.
- Much richer structure: choice, fixpoints, ...

Surprisingly, there is some clear mathematical connection.

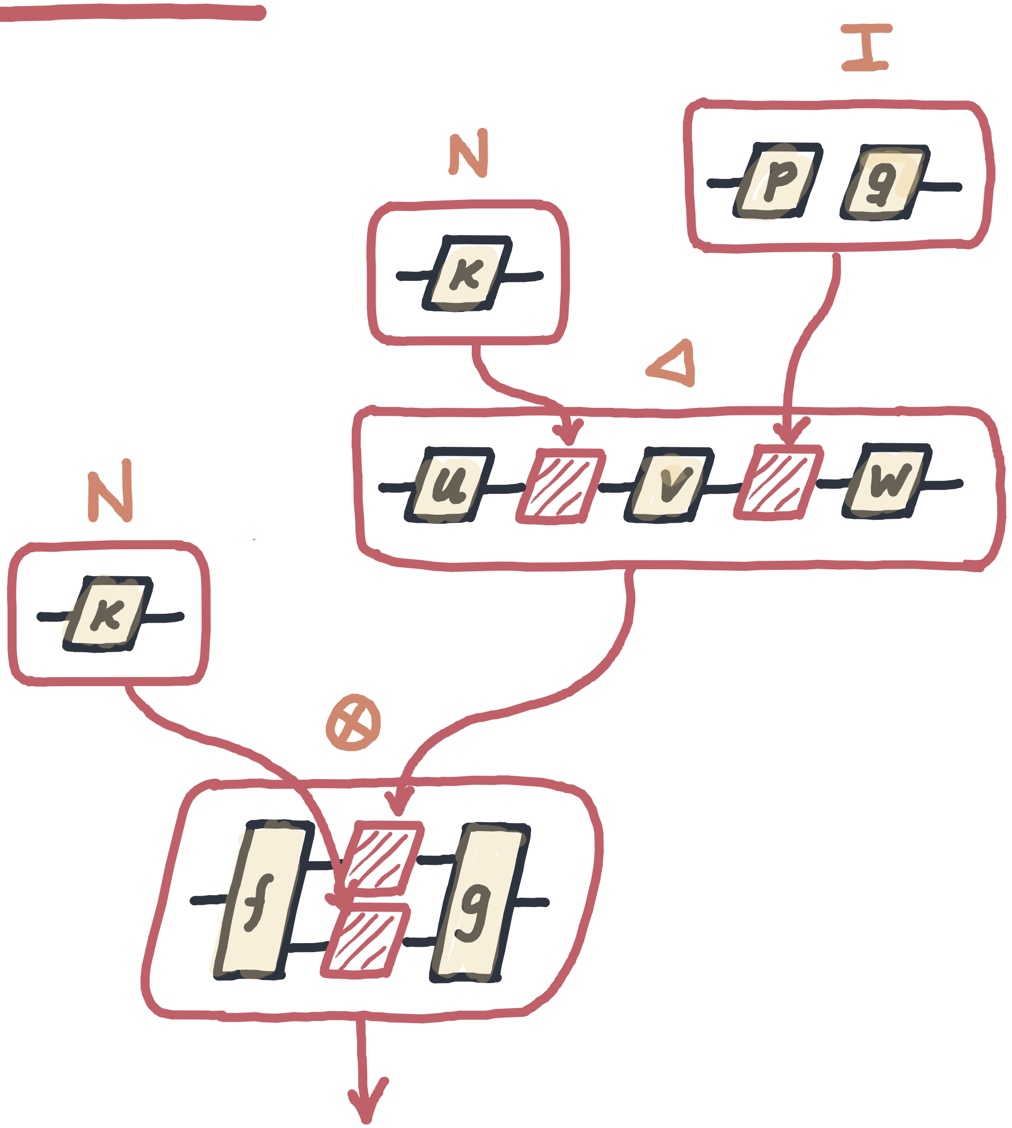
CONJECTURE. Isomix categories are normal duoidal categories.
A normal duoidal $(\mathbb{C}, *, \triangleleft, N)$ is isomix $(\mathbb{C}, \otimes = *, \parr = \triangleleft, N)$.

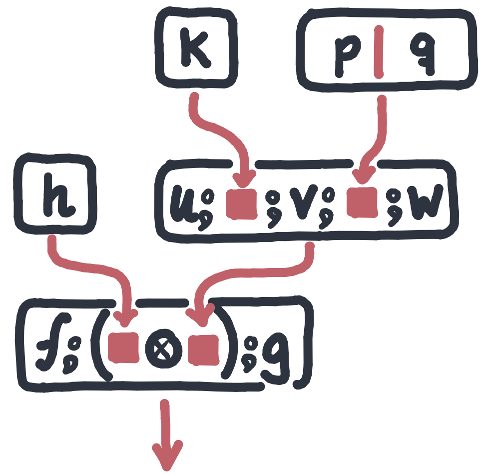
Cockett (et al); Blute, Cockett, Seely

Algebra of Lenses



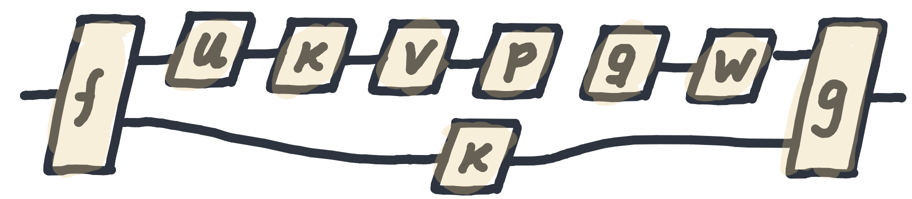
Monoidal Splice-Contour





$$= f;(h\otimes(u;k;v;p;q;w));g$$

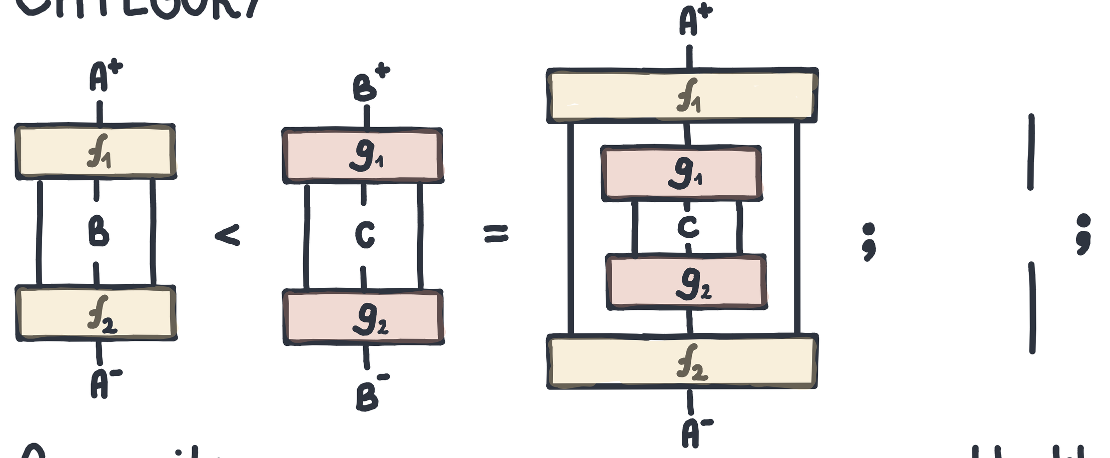
$$=$$



Produoidal Algebra of Monoidal Contexts

Earnshaw, Hefford, Román.

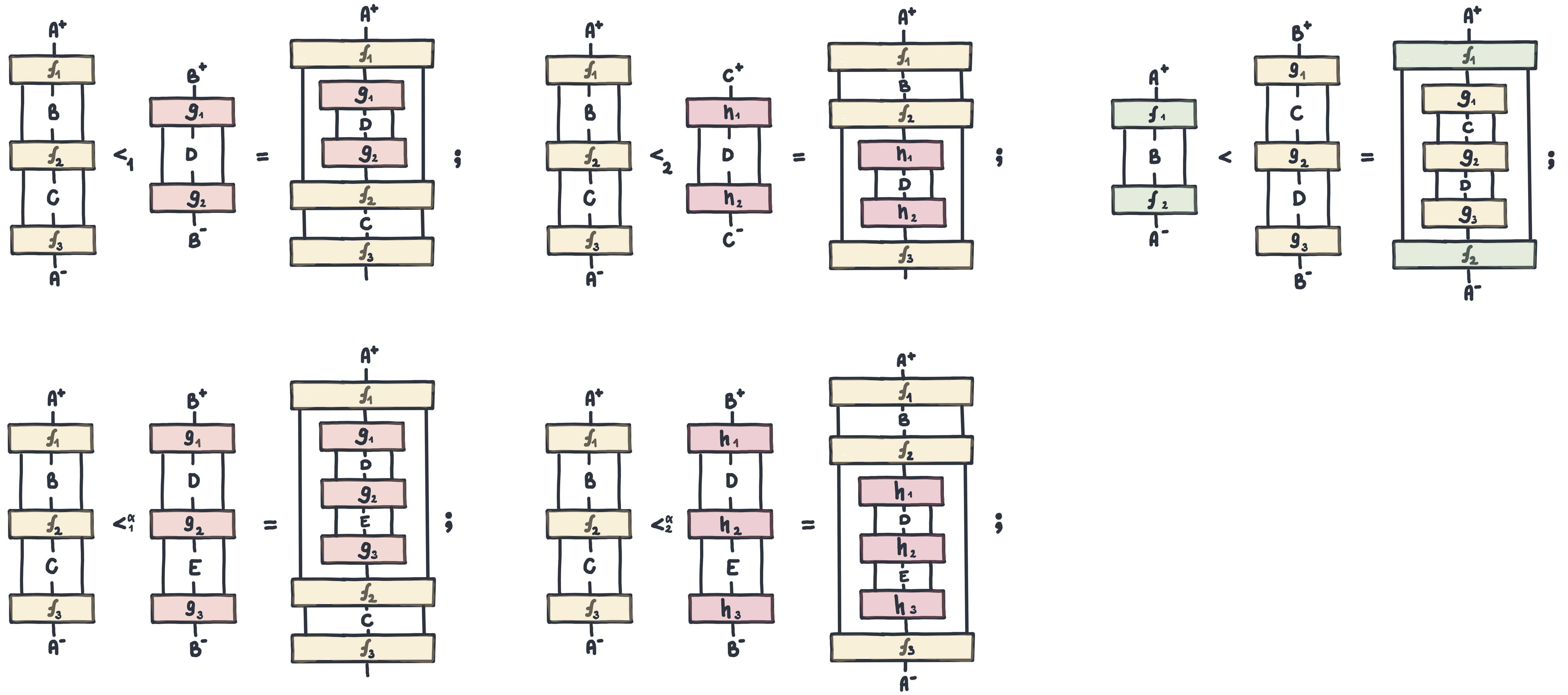
Category



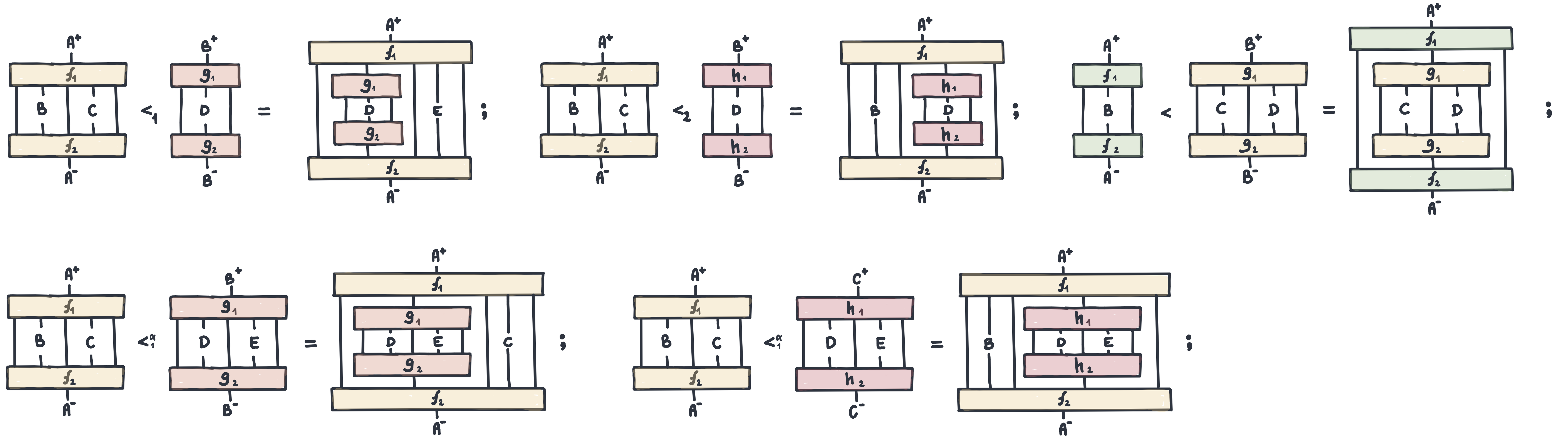
Composition

Identity

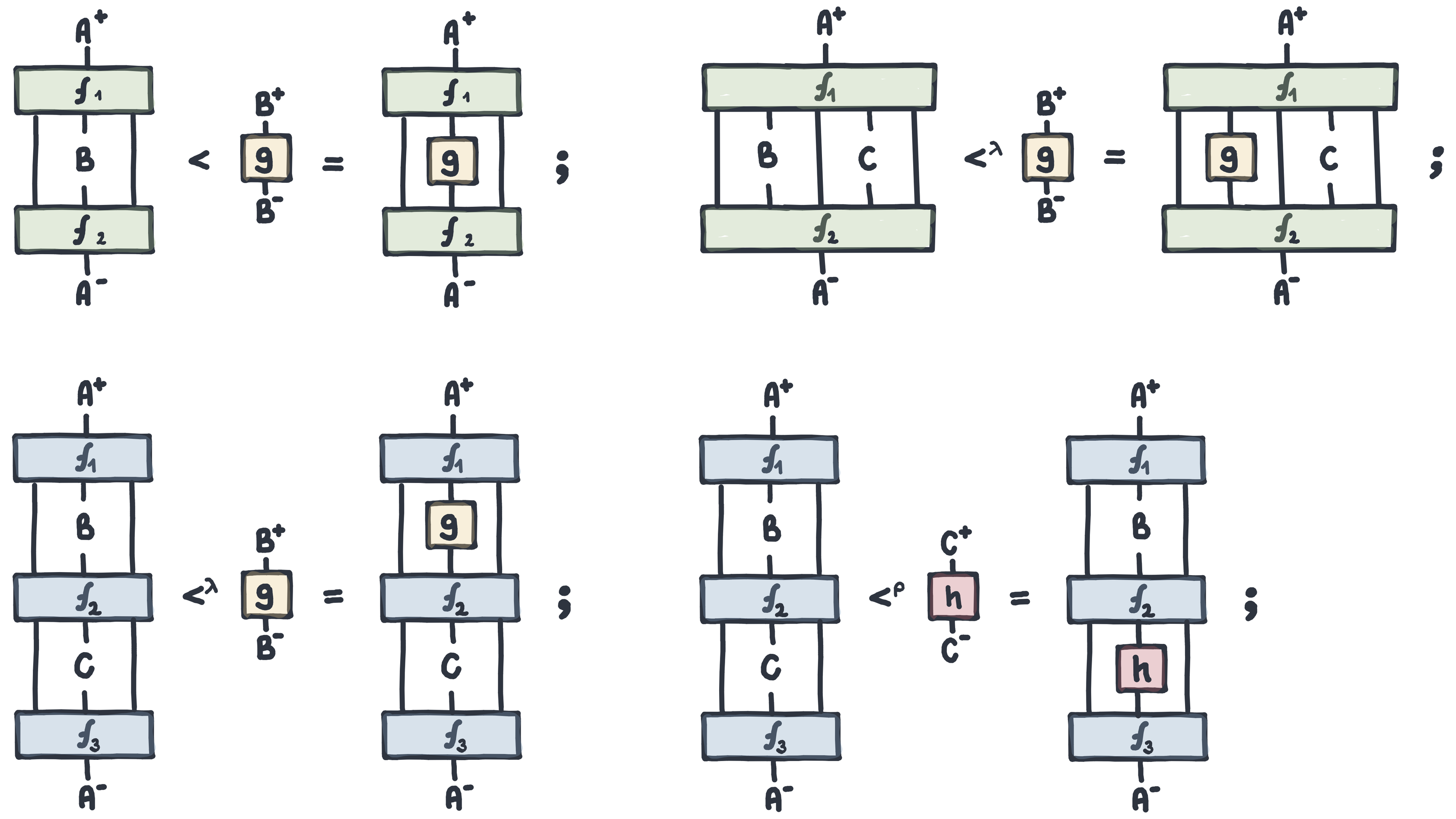
Sequential Protensor



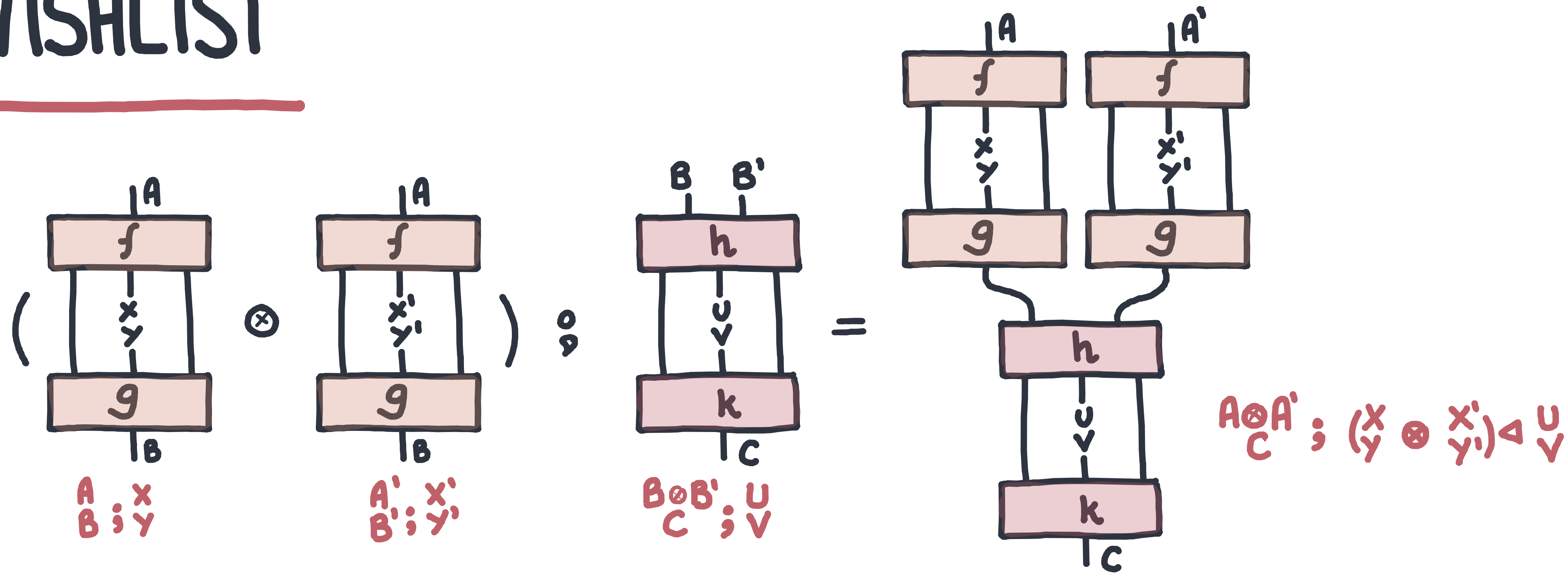
Parallel Protensor



Unit



WISHLIST



Algebra of parallel/sequential decomposition. (Pro)duoidal categories.

Semantics of incomplete diagrams. Monoidal context.

Message-passing. Send-Receive types.

Endorelations in a Monoid

Let M be any monoid. Endorelations form a duoidal poset. Commutative if M is.

$$R \triangleleft S\,(x,y) = \exists z.\, R(x,z) \wedge S(z,y), \qquad N(x,y) = (x=y),$$
$$R \otimes S\,(x,y) = \exists x_1,x_2.\,(x=x_1x_2) \wedge R(x_1,y_1) \wedge S(x_2,y_2) \wedge (y_1y_2=y), \qquad I(x,y) = (x=e=y).$$

For instance the first laxator says

$$(R \triangleleft S) \otimes (R' \triangleleft S')\,(x,y) = \exists x_1x_2y_1y_2z_1z_2.\; x = x_1x_2 \wedge \begin{matrix} R(x_1;z_1) \wedge S(z_1;y_1) \\ R(x_2;z_2) \wedge S(z_2;y_2) \end{matrix} \wedge y_1y_2 = y.$$

$$\to \exists x_1x_2y_1y_2\begin{matrix} z_1'z_2' \\ z_1z_2 \end{matrix}.\; x = x_1x_2 \wedge \begin{matrix} R(x_1;z_1) \\ R(x_2;z_2) \end{matrix} \wedge z_1z_2 = z_1'z_2' \wedge \begin{matrix} S(z_1';y_1) \\ S(z_2';y_2) \end{matrix} \wedge y_1y_2 = y.$$

Normalization gives affine relations or biaffine relations.

Profunctors

Definition. A *profunctor* $P : \mathbb{C} \nrightarrow \mathbb{D}$ is a functor
$$P : \mathbb{C}^{op} \times \mathbb{D} \to \mathsf{Set}.$$

Equivalently, a family of sets $P(X;Y)$ indexed by objects $X \in \mathbb{C}$ and $Y \in \mathbb{D}$ and with actions
$$(>) : \hom(X';X) \times P(X;Y) \to P(X';Y),$$
$$(<) : P(X;Y) \times \hom(Y;Y') \to P(X;Y'),$$
that are functorial:
$$f > (p < g) = (f > p) < g$$
$$f_0 > f_1 > p = (f_0 ⨾ f_1) > p$$

Profunctors compose via coends
$$(P ⨾ Q)(X;Z) = \int^{Y \in \mathbb{B}} P(X;Y) \times Q(Y;Z)$$
$$= \coprod_{Y \in \mathbb{B}} P(X;Y) \times Q(Y;Z) \Big/ \sim_D .$$

Normalizing Duoidals

Consider a duoidal category $(\mathbb{V}, \otimes, I, \triangleleft, N)$. If we want a normal category, we need to change $\otimes$ so that N is a unit.

- N is already a $\otimes$-monoid, with $N \otimes N \to N$ and $I \to N$.
- The category of $N^{\otimes}$-bimodules is monoidal:

$$N \otimes (A \triangleleft B) \otimes N \to (N \triangleleft N) \otimes (A \triangleleft B) \otimes (N \triangleleft N) \to (N \otimes A \otimes N) \triangleleft (N \otimes B \otimes N) \to A \triangleleft B,$$

but duoidality requires reflexive coequalizers to define the new tensor $(\otimes_N)$, and these coequalizers need to be preserved by $(\otimes)$,

$$A \otimes N \otimes B \rightrightarrows A \otimes B \twoheadrightarrow A \otimes_N B.$$

THEOREM (Garner, López Franco). Let $(\mathbb{V}, \otimes, I, \triangleleft, N)$ a duoidal with reflexive coequalizers, preserved by $(\otimes)$. Then, $(\mathrm{Bimod}_N^{\otimes}, \otimes_N, N, \triangleleft, N)$ is a normal duoidal.

Normalizing Produoidals

Normalization works always in produoidals.

Consider a produoidal category $(\mathbb{V}, \otimes, I, \triangleleft, N)$. We can ALWAYS(!) normalize it. There is a "bimodule promonad", and its category gives a normalization.

$$\mathcal{N}\mathbb{V}(X;Y) = \mathbb{V}(X; N \otimes Y \otimes N), \qquad \mathcal{N}_o\mathbb{V}(X;Y) = \mathbb{V}(X; N \otimes Y),$$

$$\mathcal{N}\mathbb{V}(X; Y \triangleleft_N Z) = \mathbb{V}(X; (N \otimes Y \otimes N) \triangleleft (N \otimes Z \otimes N)), \qquad \mathcal{N}_o\mathbb{V}(X; Y \triangleleft_N Z) = \mathbb{V}(X; (N \otimes Y) \triangleleft (N \otimes Z)),$$

$$\mathcal{N}\mathbb{V}(X; Y \otimes_N Z) = \mathbb{V}(X; N \otimes Y \otimes N \otimes Z \otimes N), \qquad \mathcal{N}_o\mathbb{V}(X; Y \otimes_N Z) = \mathbb{V}(X; N \otimes Y \otimes Z),$$

$$\mathcal{N}\mathbb{V}(X; N_N) = \mathcal{N}\mathbb{V}(X; I_N) = \mathbb{V}(X; N), \qquad \mathcal{N}_o\mathbb{V}(X; N_N) = \mathcal{N}_o\mathbb{V}(X; I_N) = \mathbb{V}(X; N).$$

THEOREM (EHR23). Let $(\mathbb{V}, \otimes, I, \triangleleft, N)$ a produoidal category. Then, $(\mathcal{N}_o\mathbb{V}, \otimes_N, \triangleleft_N, N)$ is a normal produoidal. Moreover, $\mathcal{N} : \mathrm{Produo} \to \mathrm{Produo}$.

THEOREM (EHR23). Let $(\mathbb{V}, \otimes, I, \triangleleft, N)$ a symm. produoidal category. Then, $(\mathcal{N}_o\mathbb{V}, \otimes_N, \triangleleft_N, N)$ is a normal sym produoidal. Moreover, $\mathcal{N}_o : \mathrm{Produo} \to \mathrm{Produo}$.

Normalizing Symmetric Duoidals

Consider a sym$^{\otimes}$.duoidal category $(\mathbb{V}, \otimes, I, \triangleleft, N)$. If we want a normal category, we need to change $\otimes$ so that N is a unit.

- N is already a $\otimes$-monoid, with $N \otimes N \to N$ and $I \to N$.
- The category of $N^{\otimes}$-modules is monoidal:

$$N \otimes (A \triangleleft B) \to (N \triangleleft N) \otimes (A \triangleleft B) \to (N \otimes A) \triangleleft (N \otimes B) \to A \triangleleft B,$$

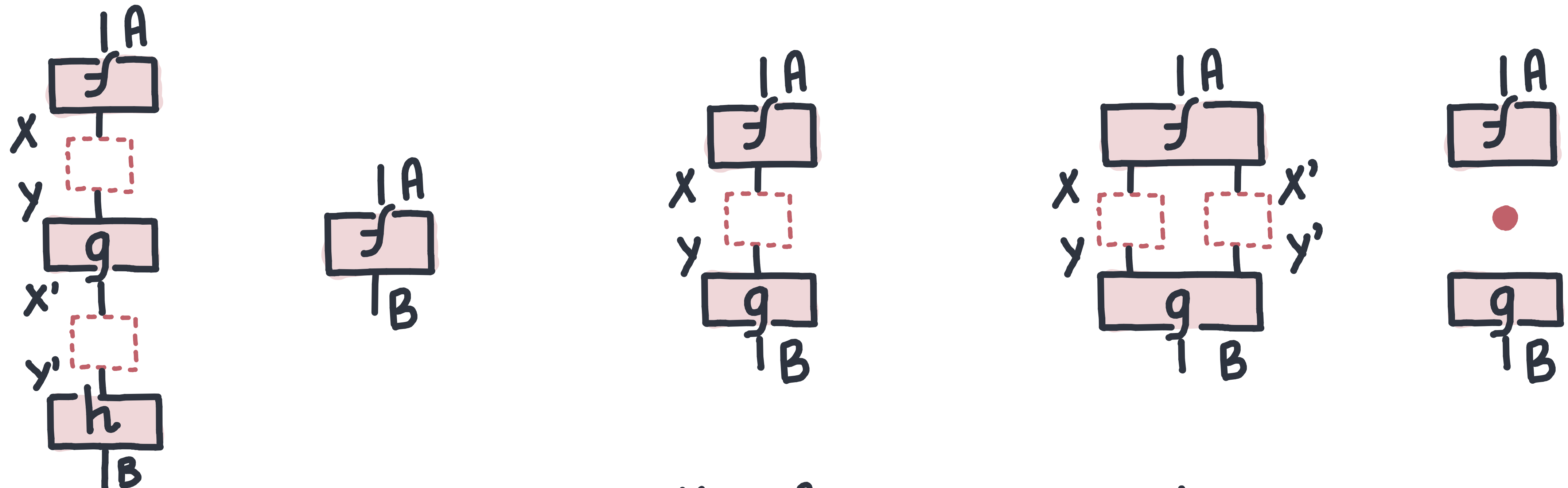
but duoidality requires reflexive coequalizers to define the new tensor $(\otimes_N)$, and these coequalizers need to be preserved by $(\otimes)$,

$$A \otimes N \otimes B \rightrightarrows A \otimes B \twoheadrightarrow A \otimes_N B .$$

THEOREM (Garner, López Franco). Let $(\mathbb{V}, \otimes, I, \triangleleft, N)$ a sym$^{\otimes}$. duoidal with reflexive coequalizers, preserved by $(\otimes)$. Then, $(\mathrm{Mod}_N^{\otimes}, \otimes_N, N, \triangleleft, N)$ is a normal sym$^{\otimes}$. duoidal.

MONOIDAL CONTEXT

THM (EHR'23). Monoidal context is the cofree produoidal on a monoidal.



$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad N \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad I \to \begin{bmatrix}A\\B\end{bmatrix}$