String Diagrams of String Diagrams

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Tallinn. Compositional Methods

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String Diagrams of String Diagrams

There has been recent interest in string diagrams of string diagrams.

- Bonchi et al. use them for bimonoidal categories;
- Zanasi et al. use them for interventions and "layered explanations";
- Vicary et al. for categorical quantum.

There are no good semantics for these. Arguably, the technology is not here yet. We would need monoidal 3-categories, with coherence conditions.

- We could impose strictness conditions.
- Instead, I want to explore this calculus from the future.

Bimodular Categories

Definition. A bimodular category is a category with left and right monoidal actions

$$(\triangleright) \colon \mathbb{M} \times \mathbb{C} \to \mathbb{C} \quad \text{and} \quad (\triangleleft) \colon \mathbb{C} \times \mathbb{N} \to \mathbb{C}$$

with coherent natural isomorphisms

$$M_1 \triangleright M_2 \triangleright X \cong (M_1 \otimes M_2) \triangleright X; \qquad X \triangleleft (N_1 \otimes N_2) \cong X \triangleleft N_1 \triangleleft N_2;$$
$$I \triangleright X \cong X; \qquad X \triangleleft I \cong X;$$
$$(M \triangleright X) \triangleleft N \cong M \triangleright (X \triangleleft N);$$

Proposition. Every monoidal category is bimodular with self-actions.

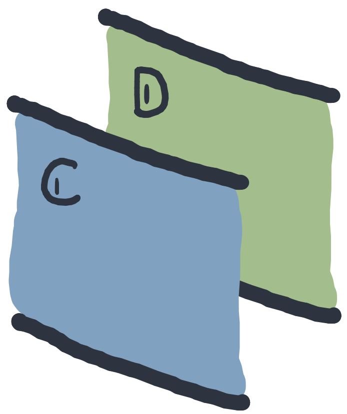
Compact Tricategory

- 0-cells are monoidal categories;
- 1-cells are pointed bimodular categories, $(\mathbb{A}^{\triangleleft}_{\triangleright}, A)$, consisting on a category $\mathbb{A}$ with two monoidal actions that are compatible; and of some object $A \in \mathbb{A}$;
- 2-cells are pointed bimod. profunctors, $T_t : (\mathbb{A}^{\triangleleft}_{\triangleright}, A) \nrightarrow (\mathbb{B}^{\triangleleft}_{\triangleright}, B)$, profunctors together with a point $t \in T(A,B)$ and compatible transformations

$$t^M_l : T(A;B) \longrightarrow T(M \triangleright A; M \triangleright B),$$
$$t^M_r : T(A;B) \longrightarrow T(A \triangleleft M; B \triangleleft M).$$

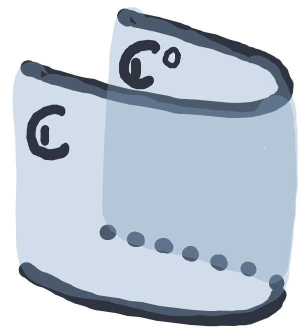
- 3-cells are homomorphisms of bimodular profunctors.

COMPACT TRICATEGORY



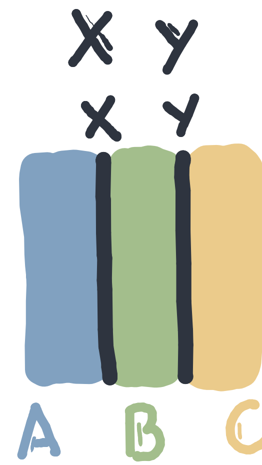
$\mathbb{C}\times\mathbb{D},\ \otimes_{\mathbb{C}}\times\otimes_{\mathbb{D}},\ I_{\mathbb{C}}\times I_{\mathbb{D}}$

Product of monoidal categories



$\mathbb{C}^{\circ},\ A\otimes^{\circ}B = B\otimes A$

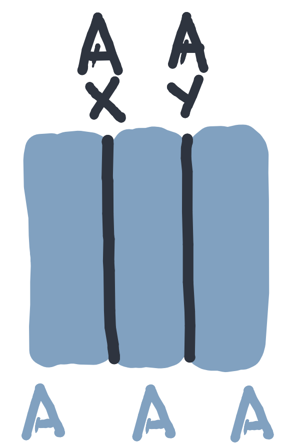
Opposite Monoidal Category



$X\otimes_{\mathbb{B}}Y,\ \triangleright_{\mathbb{A}},\ \triangleleft_{\mathbb{C}}$

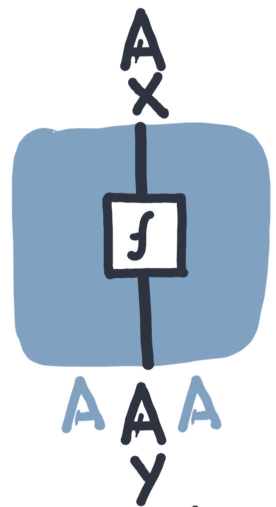
Tensor product of bimodules

Compact Tricategory



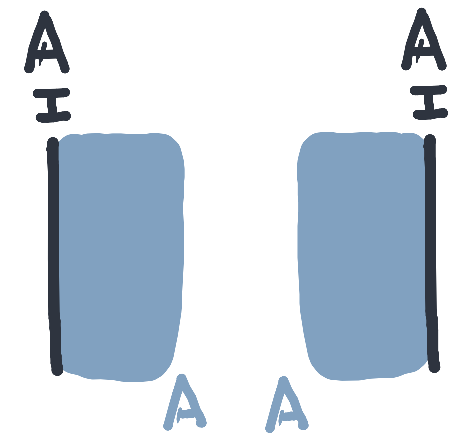
$A \otimes_A A \cong A$

Tensoring inside a monoidal category.



$f, \mathrm{hom} : A, X \nrightarrow A, Y$

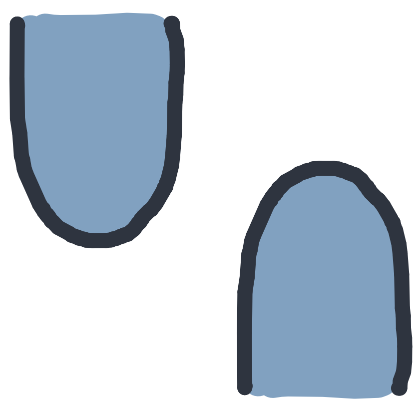
Hom is still a Tambara Module



A is a 1-bimodule

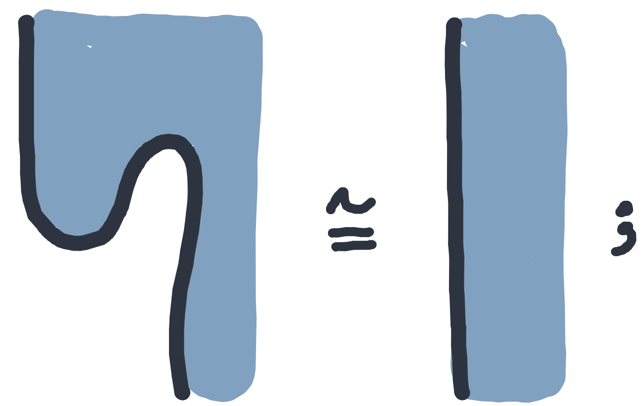
We can stop and start pieces of paper.

Compact Tricategory

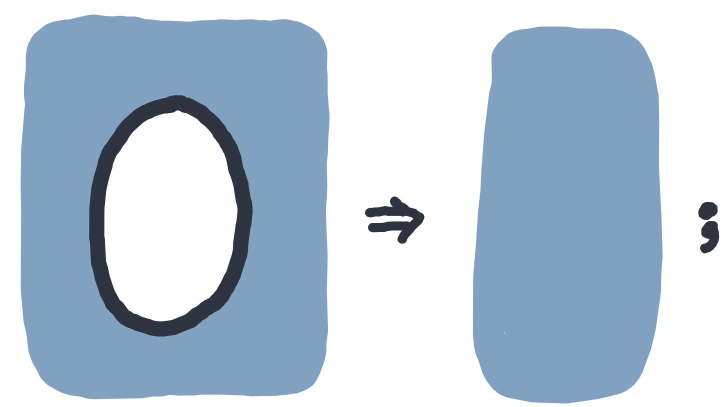


$\mathrm{id}, \hom(\cdot, I) : (\mathbb{A}, I) \nrightarrow (\mathbb{1}, *)$

Caps and cups
for internal diagrams.

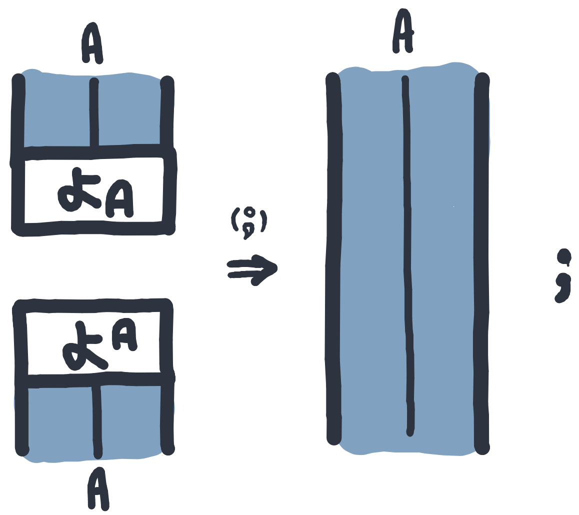


2-adjunction of
diagram borders



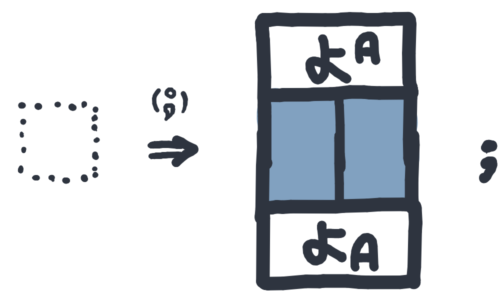
Adjunction
on tensors

Compact Tricategory

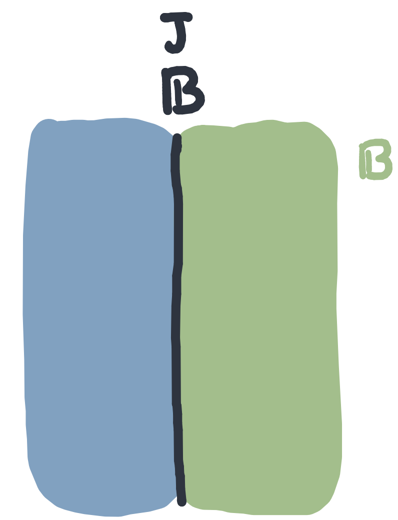


$id, よ_A : \mathbb{A}, A \nrightarrow \mathbb{1}, *$

Yoneda Embeddings, Composition



Identity function.

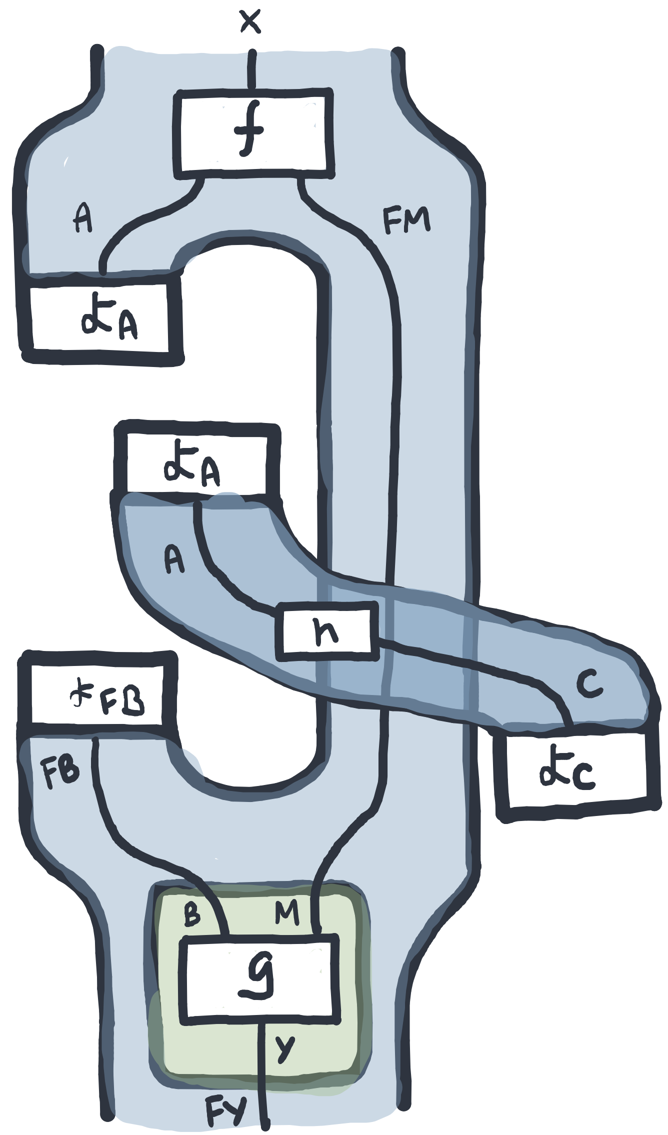


$(F\cdot \otimes \cdot) : \mathbb{A} \times \mathbb{B} \to \mathbb{B}$

$(\cdot \otimes \cdot) : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

Functor Boxes

EXAMPLE



- Bartlett, Douglas, Schommer-Pries, Vicary. Modular Categories as Representations of the 3-dimensional bordism 2-category.
- Román. Open Diagrams via Coend Calculus.