Contouring Prostar Autonomous Categories

Matt Earnshaw, James Hefford, Mario Román

Tallinn University of Technology[1,3], University of Oxford[2].

This abstract accompanies and extends our recent work "The Produoidal Algebra of Process Decomposition" [3]. There, we introduced a *splice-contour* adjunction between monoidal categories and produoidal categories, with applications to the study of multi-party protocols in computer science. This restricts to an adjunction between categories and promonoidal categories. In this abstract, we show that there is more than mere promonoidal structure, resulting in an adjunction to *prostar autonomous categories*.

1 Prostar autonomous categories

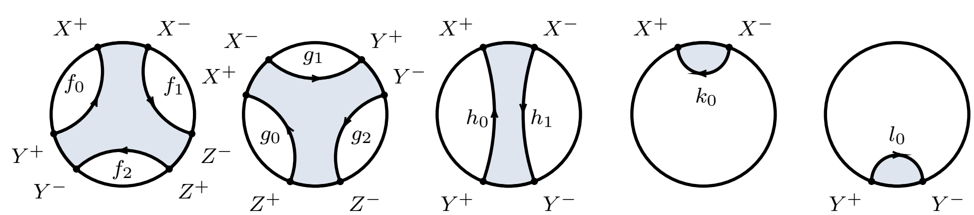
Definition 1.1. A *prostar autonomous category* is a category $\mathcal{C}$ endowed with a promonoidal structure $(\mathcal{C}, \otimes, \top)$, and procomonoidal structure $(\mathcal{C}, \parr, \bot)$, that interact as a Frobenius pseudomonoid [2, 5].[1] That is, it is a category endowed with four profunctors, suggestively written $\mathcal{C}(\bullet \otimes \bullet; \bullet)$, $\mathcal{C}(\top; \bullet)$, $\mathcal{C}(\bullet; \bot)$ and $\mathcal{C}(\bullet; \bullet \parr \bullet)$, as if they were representable. These profunctors form two promonoidal categories [1] with coherent associators and unitors. Further, they are endowed with invertible Frobenius distributors,

$$\int^W \mathcal{C}(A; C \parr W) \times \mathcal{C}(W \otimes B; D) \xrightarrow{\cong} \mathcal{C}(A \otimes B; C \parr D), \quad \int^W \mathcal{C}(A \otimes W; C) \times \mathcal{C}(B; W \parr D) \xrightarrow{\cong} \mathcal{C}(A \otimes B; C \parr D),$$

such that every formal diagram formed of these distributors and promonoidal coherences commutes.

Prostar autonomous categories have a canonical *prostar* given by profunctors $\mathcal{C}(\bullet \otimes \bullet; \bot)$ and $\mathcal{C}(\top; \bullet \parr \bullet)$. We may think of a prostar autonomous category as a category $\mathcal{C}$ equipped with sets of polymorphisms $\mathcal{C}(\bullet \otimes \ldots \otimes \bullet; \bullet \parr \ldots \parr \bullet)$. The Frobenius isomorphisms let us decompose polymorphisms into combinations of the pro(co)monoidal structures: this decomposition is unique up to dinaturality. Informally, prostar autonomous categories are to polycategories what promonoidal categories are to (co)multicategories

Definition 1.2. Let $\mathcal{C}$ be a category. Its prostar autonomous category of *spliced arrows*, $S\mathcal{C}$, has underlying category $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Intuitively, its profunctors are defined by spliced circles of morphisms.

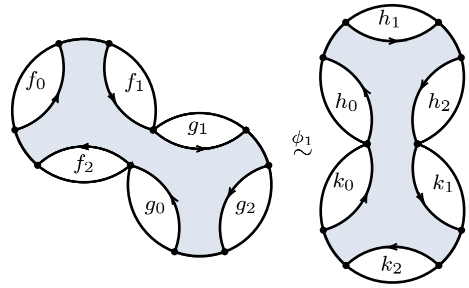


Explicitly, it is defined by the following profunctors (below, left). The coherence isomorphisms are defined by glueing circles along the desired boundary and composing the relevant arrows; two compositions are isomorphic if and only if they determine the same arrows (below, right).

Remark 1.3. This structure appeared in Day & Street [2, Ex. 7.3], where it was noticed that the canonical promonoidal category induced by a small category [1] has an involution. As a multicategory, it was rediscovered by Melliès & Zeilberger [6]. Monoidal spliced arrows were introduced by the authors [3].

[1]Street [7] proved Frobenius pseudomonoids in **Prof** to be equivalent to what Day & Street [2] called $*$-*autonomous promonoidal categories*. The minor twist "prostar autonomous" emphasizes that the canonical prostar may not be representable. When all of the structure including the prostar is representable, we obtain $*$-autonomous categories.

$$
\begin{aligned}
\mathcal{SC}\left(\tbinom{X^+}{X^-};\tbinom{Y^+}{Y^-}\parr\tbinom{Z^+}{Z^-}\right) &= \mathcal{C}(Y^+;X^+)\times\mathcal{C}(X^-;Z^-)\times\mathcal{C}(Z^+;Y^-);\\
\mathcal{SC}\left(\tbinom{X^+}{X^-}\otimes\tbinom{Y^+}{Y^-};\tbinom{Z^+}{Z^-}\right) &= \mathcal{C}(Z^+;X^+)\times\mathcal{C}(X^-;Y^+)\times\mathcal{C}(Y^-;Z^-);\\
\mathcal{SC}\left(\tbinom{X^+}{X^-};\tbinom{Y^+}{Y^-}\right) &= \mathcal{C}(Y^+;X^+)\times\mathcal{C}(X^-;Y^-);\\
\mathcal{SC}\left(\tbinom{X^+}{X^-};\bot\right) &= \mathcal{C}(X^-;X^+);\\
\mathcal{SC}\left(\top;\tbinom{Y^+}{Y^-}\right) &= \mathcal{C}(Y^+;Y^-).
\end{aligned}
$$



whenever $f_0 = k_0 \mathbin{⨾} h_0$, $f_1 \mathbin{⨾} g_1 = h_1$, $g_2 = h_2 \mathbin{⨾} k_1$, $g_0 \mathbin{⨾} f_2 = k_2$.

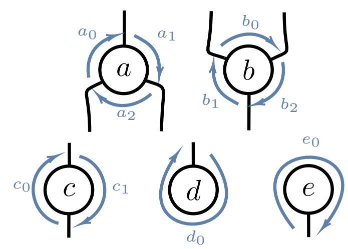
Remark 1.4. $\mathcal{SC}$ has a representable prostar, given on objects by $\binom{X^+}{X^-}^* = \binom{X^-}{X^+}$.

2 Prostar autonomous contour

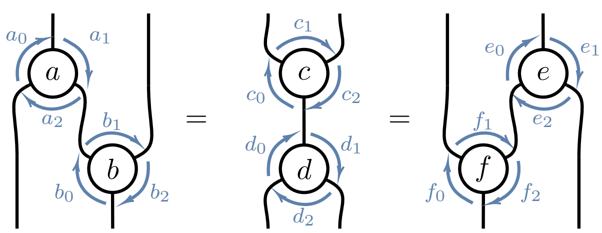
Melliès & Zeilberger [6] defined the *contour of a multicategory* as left adjoint to a *multicategory* of spliced arrows. Our work [3] extended this to promonoidal and produoidal categories. We extend this further:

Definition 2.1. The contour of a prostar autonomous category $(\mathcal{A},\otimes,\top,\parr,\bot)$ is the category $C\mathcal{A}$ having objects X^L and X^R, for each $X\in\mathcal{A}_{\text{obj}}$; and morphisms those arising from contours of decompositions of the prostar autonomous category (below, right). Specifically, it is presented by, for each:

- $a\in\mathcal{A}(X;Y\parr Z)$, a triple, $a_0\colon Y^L\to X^L$, $a_1\colon X^R\to Z^R$, $a_2\colon Z^L\to Y^R$;
- $b\in\mathcal{A}(X\otimes Y;Z)$, a triple, $b_0\colon Z^L\to X^L$, $b_1\colon X^R\to Y^L$, $b_2\colon Y^R\to Z^R$;
- $c\in\mathcal{A}(X;Y)$, a pair, $c_0\colon Y^L\to X^L$, $c_1\colon X^R\to Y^R$;
- $d\in\mathcal{A}(X;\bot)$, a morphism $d_0\colon X^R\to X^L$;
- $e\in\mathcal{A}(\top;Z)$, a morphism $e_0\colon Z^L\to Z^R$.



Apart from the quotients arising from the pro(co)monoidal structures (see [3, §3.1]), these morphisms are quotiented by equations arising from the Frobenius isomorphisms of the prostar autonomous category:



$$
\begin{aligned}
&a_0 = c_0 \mathbin{⨾} d_0 = f_0,\\
&a_1 \mathbin{⨾} b_1 = c_1 = f_1 \mathbin{⨾} e_0,\\
&b_2 = c_2 \mathbin{⨾} d_1 = e_1,\\
&b_0 \mathbin{⨾} a_2 = d_2 = e_2 \mathbin{⨾} f_2.
\end{aligned}
$$

Theorem 2.2. *Contour extends to a functor C : $\mathsf{ProStarAut}\to\boldsymbol{Cat}$, splice extends to a functor S : $\boldsymbol{Cat}\to\mathsf{ProStarAut}$, and C is left adjoint to S. The proof extends [3, Theorem 3.7].*

Remark 2.3. This work is a step towards extending the *produoidal category of spliced monoidal arrows* [3] to a structure with a more polycategorical nature. Following our previous work [3], this should permit a more flexible representation of *multi-party protocols* in computer science.

References [1] B. Day. *On closed categories of functors.* in Reports of the Midwest Category Seminar IV, volume 137, pages 1–38, Springer, 1970. [2] B. Day and R. Street. *Quantum categories, star autonomy, and quantum groupoids*, 2004. [3] M. Earnshaw, J. Hefford, and M. Román. *The produoidal algebra of process decomposition*, 2023. [4] A. D. Lauda. *Frobenius algebras and planar open string topological field theories*, 2005. [5] A. D. Lauda. *Frobenius algebras and ambidextrous adjunctions.* TAC, 16(4):84–122, 2006. [6] P. Melliès and N. Zeilberger. *Parsing as a Lifting Problem and the Chomsky-Schützenberger Representation Theorem.* in Mathematical Foundations for Programming Semantics, 2022. [7] R. Street. *Frobenius monads and pseudomonoids.* J. Math. Physics, 45(10):3930–3948, 2004.