Graded Coalgebras of Monads for Continuous Dynamics

Elena Di Lavore
University of Oxford, England

Mario Román
University of Oxford, England

Abstract

We argue for the time-graded coalgebras of probabilistic and non-determinisic monads to be suitable coalgebraic continuous-time dynamical systems.

2012 ACM Subject Classification

Keywords and phrases Coalgebra, transition systems, continuous time, category theory.

Digital Object Identifier 10.4230/LIPIcs...

1 Graded Coalgebras of Graded Monads

Coalgebras are portrayed as machines with a button and a display: each time we press the button, the machine displays a value [3, 7]. This picture is discrete: the button is a morphism, $\alpha \colon X \to F(X)$, and pressing it multiple times induces a map $X \to F^n(X)$. Grading refines this picture; and graded coalgebras, $\alpha_t \colon X \to F_t(X)$, buy us analog buttons we can modulate.

▶ **Definition 1** (Graded monad [9, 6, 1]). *A graded monad Θ in a category $\mathbb{C}$, graded on a monoid $(T, \cdot, e)$, is a family of endofunctors $\Theta_t \colon \mathbb{C} \to \mathbb{C}$ together with a family of transformations, $\mu^{s,t}_X \colon \Theta_s(\Theta_t(X)) \to \Theta_{s\cdot t}(X)$; and a natural transformation $\eta_X \colon X \to \Theta_e(X)$; making the following diagrams commute.*

$$\begin{array}{ccc} \Theta_s\Theta_t\Theta_r X & \xrightarrow{\Theta_s\mu_{t,r}} & \Theta_s\Theta_{t\cdot r}X \\ \downarrow{\scriptstyle \mu_{s,t}\Theta_r} & & \downarrow{\scriptstyle \mu_{s,t\cdot r}} \\ \Theta_{s\cdot t}\Theta_r X & \xrightarrow[\mu_{s\cdot t,r}]{} & \Theta_{s\cdot t\cdot r}X \end{array} \qquad \begin{array}{ccc} \Theta_s X & \xrightarrow{\eta\Theta_s} & \Theta_e\Theta_s X \\ & \searrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \mu_{e,s}} \\ & & \Theta_{e\cdot s}X \end{array} \qquad \begin{array}{ccc} \Theta_s X & \xrightarrow{\Theta_s\eta} & \Theta_s\Theta_e X \\ & \searrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \mu_{s,e}} \\ & & \Theta_{s\cdot e}X \end{array}$$

▶ **Definition 2** (Graded coalgebra). *A graded coalgebra for a graded monad Θ is a carrier object, X, together with a family of morphisms $\alpha_t \colon X \to \Theta_t(X)$ indexed over the monoid $(T, \cdot, e)$, and making the following diagrams commute.*

$$\begin{array}{ccc} X & \xrightarrow{\alpha_s} & \Theta_s X \\ \downarrow{\scriptstyle \alpha_{s\cdot t}} & & \downarrow{\scriptstyle \Theta_s\alpha_t} \\ \Theta_{s\cdot t}X & \xleftarrow[\mu_{s,t}]{} & \Theta_s\Theta_t X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\alpha_e} & \Theta_e X \\ \downarrow{\scriptstyle \eta} & \swarrow{\scriptstyle \mathrm{id}} & \\ \Theta_e X & & \end{array}$$

A coalgebra morphism $f \colon (X, \alpha) \to (Y, \beta)$ must be such that $\alpha_t \mathbin{⨾} f = f \mathbin{⨾} \beta_t$ for each $t \in T$.

▶ **Example 3** (The splitting interval as a list coalgebra, c.f. [8]). Lists form an $(\mathbb{N}, \cdot, 1)$-graded monad on **Set** with functors $\mathsf{List}_n(X) = X^n$ and multiplications $\mu^{m,n}_X \colon (X^n)^m \to X^{m\cdot n}$ given by flattening a list of lists. The set of closed intervals, $\mathsf{Int} = \{[x, y] \mid x, y \in \mathbb{R}\}$, is a graded coalgebra for the graded list monad. Its coalgebra morphisms, $\alpha_n \colon \mathsf{Int} \to \mathsf{List}_n(\mathsf{Int})$, map an interval $[x, y]$ to the list of intervals obtained by splitting it into n equal parts:

$$\alpha_n([x, y]) = ([z_0, z_1], [z_1, z_2], \dots, [z_{n-1}, z_n]), \text{ for } z_k = x + k \cdot \frac{y - x}{n}.$$

▶ **Proposition 4** (Coalgebras of free monads). *Coalgebras for an endofunctor are* $(\mathbb{N}, +, 0)$*-graded coalgebras for the* $(\mathbb{N}, +, 0)$*-graded monad* $F^{\circ n}$ *given by n-fold composition of the functor.*

2 Continuous-Time Dynamics

Graded coalgebras for a trivially graded monad coincide with Lawvere dynamical systems [5]: monoid homomorphisms from a monoid of grades, $(M, \cdot, e)$, to the monoid of endomorphisms in the Kleisli category, $\mathbb{C}(X; \Theta(X))$.

▶ **Example 5** (Brownian motion). The family of morphims $\beta_s \colon X \to D(X)$ defining Brownian motion, $\beta_s(x) = \mathrm{Normal}(x; s)$, form a trivially graded coalgebra for the Giry monad [2] over standard Borel spaces. The coalgebra axioms say that $y \sim \mathrm{Normal}(x; s)$ and $z \sim \mathrm{Normal}(y; t)$ imply that $z \sim \mathrm{Normal}(x; s + t)$; and that $y \sim \mathrm{Normal}(x; 0)$ implies $y = x$.

Regarding Lawvere dynamical systems as coalgebras enables more complex examples: let us translate from a family of non-deterministic snapshots, $\alpha_t(x) \in \mathcal{P}(X)$, to the set of possible paths that explain them.

We fix a monoid representing time $(T, \cdot, e)$, and an abelian group $(S, +, -, 0)$ representing a space; say, $\mathbb{R}^+$ for time and $\mathbb{R}^2$ for space. We consider the set of paths $(T \Rightarrow_0 S)$: functions from time to space, $f \colon T \to S$ starting at zero, $f(e) = 0$. In the same way that, in the discrete case, a coalgebra map translates from non-deterministic machines to stream traces, the coalgebra map in Proposition 7 can translate from continuous-time transitions to paths.

▶ **Proposition 6** (Coalgebra of paths). *The family of functions* $\beta_s \colon (T \Rightarrow_0 S) \to (T \Rightarrow_0 S) \times S$ *defined by* $\beta_t(p) = (p(t \cdot \bullet) - p(t), p(t))$ *is a* T*-graded functional coalgebra of the* S*-writer monad.*

▶ **Proposition 7** (Possible paths). *Let* $\alpha_t \colon X \to X \times S$ *be a* T*-graded relational coalgebra of the* S*-writer monad. The following relation,* $\gamma \colon X \to (T \Rightarrow_0 S)$*, defined by those paths that have a trace,* $x_\bullet \colon T \to X$*, witnessing its plausibility, is a coalgebra map:*

$$\gamma(x) = \{p \in (T \Rightarrow_0 S) \mid \exists x_\bullet \colon T \to S.(x_0 = x) \wedge \forall s, t.(x_{t \cdot s}, p(t \cdot s) - p(t)) \in \alpha_s(x_t)\}.$$

This relation maps each $x \in X$ to the set of possible paths starting from that X. This is a continuous-time analogue to computing the set of traces for a non-deterministic machine.

3 Early Idea: Stochastic Continuous Dynamics

Carefully setting up graded coalgebras allows continuous-time transitions. Apart from explicitly computing final coalgebras, two important challenges remain. Firstly, we want to force systems to depend continuously on time: this can be achieved by enriching in an appropriate category of topological spaces, as it is done for Lawvere dynamical systems. A Top-enriched graded coalgebra for a Top-monad Θ consists of a *continuous* function $\alpha \colon T \to \mathsf{C}(X, \Theta(X))$ compatible with the monad structure.

Secondly, while memoryful non-deterministic systems do not require much structure, memoryful stochastic systems seem to rely on two features particular to probabilistic programming: stochastic memoization [4] and exact observations [10]. We conjecture coalgebra may help clarifying the categorical semantics of these constructs.

References

1 Soichiro Fujii, Shin-ya Katsumata, and Paul-André Melliès. Towards a formal theory of graded monads. In Bart Jacobs and Christof Löding, editors, *Foundations of Software Science and Computation Structures - 19th International Conference, FOSSACS 2016, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings*, volume 9634 of *Lecture Notes in Computer Science*, pages 513–530. Springer, 2016. doi:10.1007/978-3-662-49630-5_30.

2 Michèle Giry. A categorical approach to probability theory. In *Categorical aspects of topology and analysis*, pages 68–85. Springer, 1982.

3 Bart Jacobs. *Introduction to coalgebra*, volume 59. Cambridge University Press, 2017.

4 Younesse Kaddar and Sam Staton. A model of stochastic memoization and name generation in probabilistic programming: categorical semantics via monads on presheaf categories. *Electronic Notes in Theoretical Informatics and Computer Science*, 3, 2023.

5 F William Lawvere. Toposes of laws of motion, 1997.

6 Stefan Milius, Dirk Pattinson, and Lutz Schröder. Generic trace semantics and graded monads. In *6th Conference on Algebra and Coalgebra in Computer Science (CALCO 2015)*, pages 253–269. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2015.

7 Dirk Pattinson. An introduction to the theory of coalgebras. *Course notes for NASSLLI*, 2003.

8 Dusko Pavlović and Vaughan Pratt. The continuum as a final coalgebra. *Theoretical Computer Science*, 280(1-2):105–122, 2002.

9 A. L. Smirnov. Graded monads and rings of polynomials. *Journal of Mathematical Sciences*, 151(3):3032–3051, 2008.

10 Dario Stein and Sam Staton. Compositional semantics for probabilistic programs with exact conditioning. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–13. IEEE, 2021.