A canonical algebra of open transition systems *

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Abstract. Feedback and state are closely interrelated concepts. Categories with feedback, originally proposed by Katis, Sabadini and Walters, are a weakening of the notion of traced monoidal categories, with several pertinent applications in computer science. The construction of the free such categories has appeared in several different contexts, and can be considered as state bootstrapping. We show that a categorical algebra for open transition systems, Span(Graph)*, also due to Katis, Sabadini and Walters, is the free category with feedback over Span(Set). Intuitively, this algebra of transition systems is obtained by adding state to an algebra of predicates, and therefore Span(Graph)* is, in this sense, the canonical such algebra.

Keywords: concurrency theory category theory transition systems \cdot feedback \cdot state \cdot algebra.

1 Introduction

State from feedback. A remarkable fact from electronic circuit design is how datastoring components can be built out of a combination of stateless components and feedback. A famous example is the (set-reset) "NOR latch": a circuit with two stable configurations that stores one bit of information.

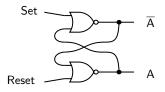


Fig. 1: NOR latch.

The NOR latch is controlled by two inputs, Set and Reset. Activating the first sets the output value to A=1; activating the second makes the output value return to A=0. This change is permanent: even when both Set and Reset are deactivated, the feedback loop maintains the last value the circuit was set to⁴—to wit, a bit of data has been conjured out

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⁴ In its original description: "the relay is designed to produce a large and **permanent** change in the current flowing in an electrical circuit by means of a small electrical stimulus received from the outside" ([11], emphasis added).

of thin air. In this paper we show that this can be seen as an instance of a more abstract phenomenon: the universal way of adding feedback to a theory of processes consists of endowing each process with a *state space*.

Indeed, there is a natural weakening of the notion of traced monoidal categories called *categories with feedback* [28]. The construction of the *free* category with feedback coincides with a "state-bootstrapping" construction, $St(\bullet)$, that appears in several different contexts in the literature [7,21,24]. We recall this construction and its mathematical status (Theorem 1), which can be summed up by the following intuition:

Theory of Processes + Feedback = Theory of Stateful Processes.

The algebra of transition systems. Our primary focus is the Span(Graph) model of concurrency, introduced in [25] as a categorical algebra of communicating state machines, or — equivalently — open transition systems. Open transition systems do not interact by input-output message passing, but by synchronization, producing a simultaneous change of state. This corresponds to a composition of spans, realized by taking a pullback in Graph. The dual algebra of Cospan(Graph) was introduced in [27]. It complements Span(Graph) by adding the operation of communicating-sequential composition [16].

Informally, a component of Span(Graph) is a state machine with states and transitions, i.e. a finite graph given by the 'head' of the span. The transition system is equipped with interfaces or *communication ports*, and every transition is labeled by the effect it produces in *all* its interfaces. We give examples below.

Stateful and stateless components. In Figure 2, we depict two open transition systems as arrows of $\mathbf{Span}(\mathbf{Graph})$. The first represents a NOR gate $\mathbb{B} \times \mathbb{B} \to \mathbb{B}$. The diagram below left is a graphical rendering of the corresponding span $\mathbb{B} \times \mathbb{B} \leftarrow N \to \mathbb{B}$, where \mathbb{B} is considered as a single-vertex graph with two edges, corresponding to the signals $\{0, 1\}$, N is the unlabeled graph depicted within the bubble, and the labels witness the action of two homomorphisms, respectively $N \to \mathbb{B} \times \mathbb{B}$ and $N \to \mathbb{B}$. Here each transition represents one of the valid input/output configurations of the gate. NOR gates are stateless components, since their transition graph N has a single vertex.

The second component is a span $L = \{\text{Set}, \text{Reset}, \text{Idle}\} \rightarrow \{A, \overline{A}\} = R$ that models a set-reset latch. The diagram below right, again, is a convenient way of denoting the relevant span $L \leftarrow D \rightarrow R$. Latches store one bit of information, they are *stateful components*; consequently, their transition graph has two states.

In both cases, the boundaries on **Span/Cospan(Graph)** are stateless: indeed, they are determined by a mere set – the self-loops of a single-vertex graph. This is a restriction that occurs rather frequently: the important subcategory of **Span(Graph)**, the one that we can clearly conceptually explain as *transition systems with interfaces*, is the full subcategory of **Span(Graph)** restricted

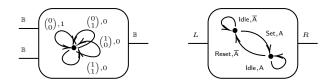


Fig. 2: A NOR gate and set-reset latch, in **Span(Graph)**.

to objects that are single-vertex graphs, which we call **Span**(**Graph**)*. Analogously, the relevant subcategory of **Cospan**(**Graph**) is **Cospan**(**Graph**)*, the full subcategory on sets, or graphs with an empty set of edges.

Definition. **Span**(**Graph**) $_*$ is the full subcategory of **Span**(**Graph**) with objects the single-vertex graphs.

The problem with $Span(Graph)_*$ is that it is—at first sight—somewhat mysterious from the categorical point of view; the morphisms are graphs, but the boundaries are given by sets. *Decorated* and *structured* spans and cospans [13,3] were introduced as theoretical frameworks to capture such phenomena, which occur frequently when composing network structures. Nevertheless, they do not quite answer the question of *why* such examples do arise naturally.

Canonicity and our original contribution. Universal constructions, such as "state-bootstrapping" $St(\bullet)$, characterize the object of interest up to equivalence, making it the canonical object satisfying some properties. This is the key to avoiding the problem outlined by Abramsky [1]: because of the lack of consensus about the intrinsic primitives of concurrency, we risk making our results too dependent on a specific syntax. It is thus important to characterize existing modeling formalisms for concurrent systems in terms of universal properties.

The main contribution of this paper is the characterization of $\mathbf{Span}(\mathbf{Graph})_*$ in terms of a universal property: it is equivalent to the free category with feedback over the category of spans of functions. We now state this more formally:

Theorem. The free category with feedback over **Span(Set)** is equivalent to **Span(Graph)***, the full subcategory of **Span(Graph)** given by single-vertex graphs. That is, there is an equivalence of categories

$$St(Span(Set)) \cong Span(Graph)_*$$
.

Given that **Span(Set)**, the category of spans of functions, can be considered an *algebra of predicates* [4,10], the high level intuition that summarizes our main contribution (Theorem 2) can be given as:

Algebra of Predicates + Feedback = Algebra of Transition Systems.

We similarly prove (Section 3.4) that the free category with feedback over **Cospan(Set**) is equivalent to **Cospan(Graph)***, the full subcategory on discrete graphs of **Cospan(Graph)**.

Related Work. Span/Cospan(Graph) has been extensively used for the modeling of concurrent systems [25,27,39,40,9,36,16,14,15]. Similar approaches to compositional modeling of networks have used *decorated* and *structured cospans* [13,3]. Despite this, Span(Graph)* has not previously been characterized in terms of a universal property.

In [28], the $St(\bullet)$ construction (under a different name) is exhibited as the free category with feedback. Categories with feedback have been arguably underappreciated but, at the same time, the $St(\bullet)$ construction has made multiple appearances as a "state bootstrapping" technique across the literature. The $St(\bullet)$ construction is used to describe a string diagrammatic syntax for concurrency theory in [7]; a variant of it had been previously applied in the setting of cartesian bicategories in [24]; and it was again rediscovered to describe a memoryful geometry of interaction in [21]. However, a coherent account of both categories with feedback and their relation with these stateful extensions has not previously appeared. This motivates our extensive preliminaries in Sections 2.1 and 2.2.

Synopsis. Section 2 contains preliminary discussions on traced monoidal categories and categories with feedback; it explicitly describes St(•), the free category with feedback. It collects mainly expository material. Section 3 exhibits a universal property for the Span(Graph)∗ and Cospan(Graph)∗ models of concurrency and Section 3.5 discusses a specific application.

Conventions. We write composition of morphisms in diagrammatic order, (f;g). When describing morphisms in a symmetric monoidal category whose input and output are known, we omit the associators and unitors, implicitly using the coherence theorem for monoidal categories.

2 Preliminaries: categories with feedback

Categories with feedback were introduced in [28], and motivated by examples such as *Elgot automata* [12], *iteration theories* [6] and *continuous dynamical systems* [26]. We recall their definition below, contrast them with the stronger notion of *traced monoidal categories* in Section 2.2, discuss the relationship between feedback and delay in Section 2.3, recall the construction of a free category with feedback in Section 2.4 and conclude with some examples in Section 2.5.

2.1 Categories with feedback

A feedback operator, $fbk(\bullet)$, takes a morphism $S \otimes A \to S \otimes B$ and "feeds back" one of its outputs to one of its inputs of the same type, yielding a morphism $A \to B$ (Figure 3, left). When using string diagrams, we depict the action of the feedback operator as a loop with a double arrowtip (Figure 3, right).

$$\frac{f \colon S \otimes A \to S \otimes B}{\mathsf{fbk}_S(f) \colon A \to B} \qquad \qquad S \qquad \qquad f \qquad \qquad$$

Fig. 3: Type and graphical notation for the operator $\mathsf{fbk}_S(\bullet)$.

Capturing a reasonable notion of feedback requires the operator to interact nicely with the flow imposed by the structure of a symmetric monoidal category. This interaction is expressed by a few straightforward axioms.

Definition 1. A category with feedback [28] is a symmetric monoidal category C endowed with an operator

$$\mathsf{fbk}_S \colon \mathbf{C}(S \otimes A, S \otimes B) \to \mathbf{C}(A, B),$$

which satisfies the following axioms (A1-A5, see also Figure 4).

- (A1). Tightening, u; $\mathsf{fbk}_S(f)$; $v = \mathsf{fbk}_S((\mathrm{id} \otimes u); f; (\mathrm{id} \otimes v))$.
- (A2). Vanishing, $fbk_I(f) = f$.
- $\begin{array}{ll} (A3). \ \ \text{Joining, fbk}_T(\mathsf{fbk}_S(f)) = \mathsf{fbk}_{S\otimes T}(f). \\ (A4). \ \ \text{Strength, fbk}_S(f)\otimes g = \mathsf{fbk}_S(f\otimes g). \end{array}$
- (A5). Sliding, $\operatorname{fbk}_T(f;(h\otimes\operatorname{id}))=\operatorname{fbk}_S((h\otimes\operatorname{id});f)$, for $h\colon S\to T$ any isomorphism.

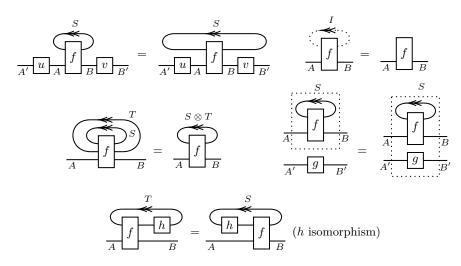


Fig. 4: Diagrammatic depiction of the axioms of feedback.

The natural notion of homomorphism between categories with feedback is that of a symmetric monoidal functor that moreover preserves the feedback structure. These are called *feedback functors*.

Definition 2. A feedback functor $F: \mathbb{C} \to \mathbb{D}$ between two categories with feedback (C, fbk^C) and (D, fbk^D) is a strong symmetric monoidal functor such that

$$F(\operatorname{fbk}_S^{\mathbf{C}}(f)) = \operatorname{fbk}_{F(S)}^{\mathbf{D}}(\mu; Ff; \mu^{-1}),$$

where $\mu_{A,B} \colon F(A) \otimes F(B) \to F(A \otimes B)$ is the structure morphism of the strong monoidal functor F. We call Feedback to the category of (small) categories with feedback and feedback functors between them. There exists a forgetful functor $\mathcal{U} \colon \mathsf{Feedback} \to \mathsf{SymMon}$.

2.2 Traced monoidal categories

Categories with feedback are a weakening of the well known traced monoidal categories. Between them, there is an intermediate notion called *right traced category* [37] that strengthens the sliding axiom from isomorphisms to arbitrary morphisms. This first extension would be already too strong for our purposes later in Section 2.4: we would be unable to define a *state space* up to isomorphism. However, the more conceptual difference of traced monoidal categories is the "yanking axiom" (in Figure 5). Indeed, strengthening the sliding axiom and adding the yanking axiom yields the definition of traced monoidal category.

Traced monoidal categories are widely used in theoretical computer science. Since their conception [22] as an abstraction of the trace of a matrix in linear algebra, they have been used in linear logic and geometry of interaction [1,17,18], programming language semantics [19], automata theory [2] and fixed point operators [20,5].

Traces are thus undeniably important, but it is questionable whether we really want to always impose *all* of their axioms. Specifically, we will be concerned with the *yanking axiom* that states that $tr(\sigma) = id$. The yanking axiom is incontestably elegant from the

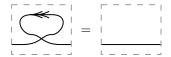


Fig. 5: The yanking axiom.

geometrical point of view: strings are "pulled", and feedback (depicted as a loop with an arrowtip) disappears (Figure 5). However, if feedback can disappear without leaving any imprint, that must mean that it is *instantaneous*: its output necessarily mirrors its input.⁵ Importantly for our purposes, this seems to imply that a feedback satisfying the yanking equation is "memoryless", or "stateless".

Consider again the NOR latch from Figure 1. We have seen how to model NOR gates in **Span(Graph)** in Figure 2, and the algebra of **Span(Graph)** does include a trace (Figure 6). However, imitating the real-world behavior of the NOR latch with *just* a trace is unsatisfactory: the trace of **Span(Graph)** is built out of stateless components, and tracing stateless components yields again a stateless component.

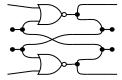


Fig. 6: Diagram for the NOR latch, modeled with a trace in **Span(Graph)**.

In engineering and computer science, instantaneous feedback is actually a rare concept; a more common notion is that of *guarded feedback*. Consider *signal*

⁵ In other words, traces are used to talk about processes in *equilibrium*, processes that have reached a *fixed point*. A theorem by Hasegawa [20] and Hyland [5] corroborates this interpretation: a trace in a cartesian category corresponds to a *fixpoint operator*.

flow graphs [38,32]: their categorical interpretation in [8] models feedback not by the usual trace, but by a trace "guarded by a register", that delays the signal and violates the yanking axiom (see Remark 7.8 in loc.cit.).

The component that trace misses in such examples is a delay.

2.3 Delay and feedback

The main difference between categories with feedback and traced monoidal categories is the failure of the yanking axiom. Consider the process that only "feeds back" its own input to itself then uses the "fed back" input to produce its output. We call this pro-

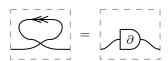


Fig. 7: Definition of delay.

cess, $\partial_A := \mathsf{fbk}_A^{\mathbf{C}}(\sigma_{A,A})$, the delay endomorphism. The yanking axiom of traced monoidal categories states that the delay is equal to the identity, which is not necessarily true for categories with feedback. In that sense, a non-trivial delay is what sets apart categories with feedback from traced monoidal categories.

This interpretation of feedback as the combination of trace and delay can be made into a theorem when the category has enough structure. Compact closed categories are traced monoidal categories where every object A has a dual A^* and the trace is constructed from two pieces $\varepsilon \colon A \otimes A^* \to I$ and $\eta \colon I \to A^* \otimes A$. Even if not every traced monoidal category is compact closed, it is true that every traced monoidal category embeds fully faithfully into a compact closed category.⁶ In a compact closed category, a feedback operator is necessarily a trace "guarded" by a delay.

Proposition 1 (Feedback from delay, [7]). Let C be a compact closed category with fbk^C a feedback operator that takes a morphism $S \otimes A \to S \otimes B$ to a morphism $A \to B$, satisfying the axioms of feedback (that we saw in Figure 4) but possibly failing to satisfy the yanking axiom of traced monoidal categories. Then the feedback operator is necessarily of the form

$$\mathsf{fbk}_S^{\mathbf{C}}(f) \coloneqq (\varepsilon \otimes \mathrm{id}); (\mathrm{id} \otimes f); (\mathrm{id} \otimes \partial_S \otimes \mathrm{id}); (\eta \otimes \mathrm{id})$$

where $\partial_A \colon A \to A$ is a family of endomorphisms satisfying

- $-\partial_A \otimes \partial_B = \partial_{A \otimes B}$ and $\partial_I = id$, and
- $-\partial_A$; h = h; ∂_B for each isomorphism h: $A \cong B$.

In fact, any family of morphisms ∂_A satisfying these properties determines uniquely a feedback operator that has ∂_A as its delay endomorphisms.

⁶ This is the **Int** construction from [22].

Proof. Given a family ∂_S satisfying the two properties, we can define a feedback structure to be $\mathsf{fbk}_S^{\mathbf{C}}(f) \coloneqq (\varepsilon \otimes \mathrm{id}); (\mathrm{id} \otimes f); (\mathrm{id} \otimes \partial_S \otimes \mathrm{id}); (\eta \otimes \mathrm{id})$ and check that it satisfies all the axioms of feedback (Figure 4). Note here that, as expected, the yanking equation is satisfied precisely when delay endomorphisms are identities, $\partial_A = \mathrm{id}_A$.



Fig. 8: Feedback from delay.

Let us now show that any feedback operator in a compact closed category is of this form. Indeed,

$$\begin{split} \mathsf{fbk}_S^\mathbf{C}(f) &= \mathsf{fbk}_S^\mathbf{C}((\mathrm{id} \otimes \varepsilon \otimes \varepsilon \otimes \mathrm{id}); (\sigma \otimes \sigma \otimes f); (\mathrm{id} \otimes \eta \otimes \eta \otimes \mathrm{id})) \\ &= (\mathrm{id} \otimes \varepsilon \otimes \varepsilon \otimes \mathrm{id}); (\mathsf{fbk}_S^\mathbf{C}(\sigma) \otimes \sigma \otimes f); (\mathrm{id} \otimes \eta \otimes \eta \otimes \mathrm{id}) \\ &= (\varepsilon \otimes \mathrm{id}); (\mathrm{id} \otimes f); (\mathrm{id} \otimes \mathsf{fbk}_S^\mathbf{C}(\sigma) \otimes \mathrm{id}); (\eta \otimes \mathrm{id}). \end{split}$$

Here we have used the fact that the trace is constructed by two separate pieces: ε and η ; and then the fact that the feedback operator, like trace, can be applied "locally" (see the axioms in Figure 4).

Consider one more time the NOR latch from Figure 1. The algebra of **Span(Graph)** does also include a feedback operator that is *not* a trace. This feedback operator is indeed canonical, in that it is the one that makes **Span(Graph)** the canonical category with feedback containing spans of functions. Imitating the real-world behavior of the NOR

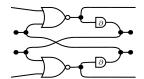


Fig. 9: NOR latch with feedback.

latch is finally possible: one of the components that builds up this feedback (and in fact, the only difference with the previous trace) is a stateful delay component. The emergence of state from feedback is witnessed by the $\mathsf{St}(ullet)$ construction.

2.4 St(•), the free category with feedback

In this section, we identify the construction that yields the free category with feedback over a symmetric monoidal category. The $St(\bullet)$ construction is a general way of endowing a system with state. It appears multiple times across the literature in slightly different forms: it constructs a stateful resource calculus in [7]; a variant is used for geometry of interaction in [21]; it coincides with the free category with feedback presented in [28]; and yet another, slightly different formulation was given in [24].

Definition 3 (Category of stateful processes, [28]). Let (\mathbf{C}, \otimes, I) be a symmetric monoidal category. We call $\mathsf{St}(\mathbf{C})$ to the category having the same objects as \mathbf{C} but where morphisms $A \to B$ are pairs $(S \mid f)$, consisting of a state space $S \in \mathbf{C}$ and a morphism $f \colon S \otimes A \to S \otimes B$. We consider morphisms up to isomorphism classes of their state space, and thus

$$(S \mid f) = (T \mid (h^{-1} \otimes id); f; (h \otimes id)),$$
 for any isomorphism $h: S \cong T$.

When depicting a stateful process, we explicitly mark the strings forming the space state. That is, an equivalence class will be depicted as any of its representatives plus some strings marked.

$$\frac{S}{A} f = \frac{T}{A} f h$$

Fig. 10: We depict stateful processes by marking the space state.

We define the identity stateful process on $A \in \mathbb{C}$ as $(I \mid \mathrm{id}_{I \otimes A})$. Sequential composition of the two stateful processes $(S \mid f) \colon A \to B$ and $(T \mid g) \colon B \to C$ is defined by

$$(S \mid f); (T \mid g) = (S \otimes T \mid (\sigma \otimes id); (id \otimes f); (\sigma \otimes id); (id \otimes g)).$$

Parallel composition of the two stateful processes $(S \mid f): A \to B$ and $(S' \mid f'): A' \to B'$ is defined by

$$(S \mid f) \otimes (S' \mid f') = (S \otimes S' \mid (\mathrm{id} \otimes \sigma \otimes \mathrm{id}); (f \otimes f'); (\mathrm{id} \otimes \sigma \otimes \mathrm{id})).$$

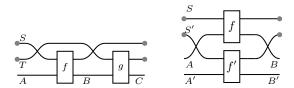


Fig. 11: Sequential and parallel composition of stateful processes.

This defines a symmetric monoidal category. Moreover, it is a category with feedback with the operator

$$\mathsf{store}_T(S \mid f) \coloneqq (S \otimes T \mid f).$$

Theorem 1. [28] The category $(St(C), store(\bullet))$ is the free category with feedback over a symmetric monoidal category C.

Fig. 12: The store(•) operation, in diagrammatic terms.

2.5 Examples of categories with feedback

Our first source of examples is *traced monoidal categories*. The axioms of feedback are a strict weakening of the axioms of trace, and every traced category is automatically a category with feedback. A more interesting source of examples is the $\mathsf{St}(ullet)$ construction we just defined.

Example 1. Consider $\mathsf{St}(\mathbf{Set})$, the free category with feedback over the monoidal structure of sets with the cartesian product. A *Mealy (or deterministic) transition system* with boundaries A and B, and state space S was originally defined [33, §2.1] to be just a function $f: S \times A \to S \times B$, which is a morphism of $\mathsf{St}(\mathbf{Set})$ up to isomorphism of the state space. Mealy transitions compose sequentially and in parallel following Definition 3, and they form a category with feedback $\mathsf{Mealy} := \mathsf{St}(\mathbf{Set})$.

$$\mathsf{fbk}\left(\begin{array}{c}0,1/1\\0\\0,0/0\end{array}\right) = \left(\begin{array}{c}0\\0\\1\end{array}\right)$$

Fig. 13: Feedback of a Mealy transition system.

The feedback operator of Mealy transitions *internalizes* input/output pairs as states. Figure 13 is an example.

It is traditional to depict automata as state/transition graphs. The characterization $\mathbf{Span}(\mathbf{Graph})_* \cong \mathsf{St}(\mathbf{Span}(\mathbf{Set}))$ that we prove in Section 3 lifts the inclusion $\mathbf{Set} \to \mathbf{Span}(\mathbf{Set})$ to a feedback functor $\mathbf{Mealy} \to \mathbf{Span}(\mathbf{Graph})_*$. This inclusion embeds a deterministic transition system into the graph that determines it.

Similarly, when we consider **Set** to be the monoidal structure of sets with the disjoint union, the notion we recover is that of an *Elgot automaton* [12], given by a transition function $S + A \rightarrow S + B$. These categories of transition systems motivate the work in [24,28].

Example 2. A linear dynamical system with inputs in \mathbb{R}^n , outputs in \mathbb{R}^m and state space in \mathbb{R}^k is given by a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Mat}_{\mathbb{R}}(k+m,k+n)$ [23]. Two linear dynamical systems $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\begin{pmatrix} A' & B' \\ C' & D \end{pmatrix}$ are considered equal whenever there is an invertible matrix $H \in \mathbf{Mat}_{\mathbb{R}}(k,k)$ such that

$$A' = H^{-1}AH$$
, $B' = BH$, $C' = H^{-1}C$.

Linear dynamical systems are morphisms of a category with feedback which coincides with $St(\mathbf{Vect}^{\oplus}_{\mathbb{R}})$. The feedback operator is defined by

$$\mathsf{fbk}_l(k, \left(\begin{smallmatrix} A_1 & A_2 & B_1 \\ A_3 & A_4 & B_2 \\ C_1 & C_2 & D \end{smallmatrix}\right)) = (k + l, \left(\begin{smallmatrix} A_1 & A_2 & B_1 \\ A_3 & A_4 & B_2 \\ C_1 & C_2 & D \end{smallmatrix}\right))$$

where
$$\begin{pmatrix} A_1 & A_2 & B_1 \\ A_3 & A_4 & B_2 \\ C_1 & C_2 & D \end{pmatrix} \in \mathbf{Mat}_{\mathbb{R}}(k+l+m,k+l+n).$$

3 Span(Graph): an algebra of transition systems

Span(Graph) [25] is an algebra of "open transition systems". It has applications in *concurrency theory* and *verification* [24,25,27,29,16], but it has also been

recently applied to biological systems [14,15]. Just as ordinary Petri nets have an underlying (firing) semantics in terms of transition systems, **Span(Graph)** is used as a semantic universe for a variant of open Petri nets, see [40,9].

An open transition system is a morphism of **Span(Graph)**: it consists of a graph endowed with two boundaries or communication ports; each transition of the graph has an effect on each boundary, and this data is used to synchronize a network of multiple transition systems. This conceptual picture actually describes a subcategory, **Span(Graph)***, where boundaries are described by mere sets, accounting for the alphabets of signals that open transition systems synchronize on. In this section we recall the details of **Span(Graph)*** and show that it is universal in the following sense:

Span(Graph)_∗ is the free category with feedback over Span(Set).

3.1 The algebra of Span(Graph).

Definition 4. A span [4,10] from A to B, both objects of a category \mathbb{C} , is a pair of morphisms with a common domain, $A \leftarrow E \rightarrow B$. The object E is the "head" of the span, and the morphisms are the left and right "legs", respectively.

When the category \mathbf{C} has pullbacks, we can sequentially compose two spans $A \leftarrow E \rightarrow B$ and $B \leftarrow F \rightarrow C$ into a span $A \leftarrow E \times_B F \rightarrow C$. Here, $E \times_B F$ is the pullback of E and F along B: for instance, in the category \mathbf{Set} of sets and functions, $E \times_B F$ is the subset of $E \times F$ given by pairs whose two components have the same image on B.

Definition 5. Let \mathbf{C} be a category with pullbacks. **Span**(\mathbf{C}) is the category that has the same objects as \mathbf{C} and isomorphism classes of spans between them as morphisms. That is, two spans are considered equal if there is an isomorphism between their heads that commutes with both legs. Dually, let \mathbf{C} be a category with pushouts. **Cospan**(\mathbf{C}) is the category **Span**(\mathbf{C}^{op}).

Span(**C**) is a symmetric monoidal category when **C** has products. The parallel composition of $A \leftarrow E \rightarrow B$ and $A' \leftarrow E' \rightarrow B'$ is given by the componentwise product $A \times A' \leftarrow E \times E' \rightarrow B \times B'$. An example is again **Span**(**Set**).

Definition 6. The category **Graph** of graphs has graphs $G = (s, t: E \Rightarrow V)$ as objects. A morphism $G \to G'$ in this category is given by two functions $e: E \to E'$ and $v: V \to V'$ such that e; s' = s; v and e; t' = t; v. In other words, it is the presheaf category on the diagram $(\bullet \Rightarrow \bullet)$.

Recall, however, that we are not interested in the whole **Span(Graph)** but only in **Span(Graph)**_{*}, the spans of graphs that have a graph of the form $A \Rightarrow 1$ on the boundaries.

Definition 7. An open transition system, a morphism of $Span(Graph)_*$, is a span of sets $A \leftarrow E \rightarrow B$ where the head is the set of transitions of a graph $E \rightrightarrows$

$$\begin{array}{cccc} A \xleftarrow{a} & E & \xrightarrow{b} B \\ \left(\begin{array}{c} \\ \end{array} \right) & \stackrel{s}{\sim} \left(\begin{array}{c} \\ \end{array} \right)^t & \left(\begin{array}{c} \\ \end{array} \right) \\ 1 \longleftarrow V \longrightarrow 1 \end{array}$$

Fig. 14: A morphism of **Span(Graph)**_{*}.

V. Two open transition systems are considered equal if there is an isomorphism between their graphs that commutes with the legs of the span.

We have been calling "stateless" to the open transition systems whose graph $E \rightrightarrows V$ has a single vertex, V = 1.

Sequential composition (the communicating-parallel operation of [25]) of two open transition systems with spans $A \leftarrow E \rightarrow B$ and $B \leftarrow F \rightarrow C$ and graphs $E \rightrightarrows S$ and $F \rightrightarrows T$ yields the open transition system with span $A \leftarrow E \times_B F \rightarrow C$ and graph $E \times_B F \rightrightarrows S \times T$. This means that the only allowed transitions are those that synchronize E and F on the common boundary B.

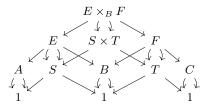


Fig. 15: Sequential composition in $Span(Graph)_*$.

Parallel composition (the non communicating-parallel operation of [25]) of two open transition systems with spans $A \leftarrow E \rightarrow B$ and $A' \leftarrow E' \rightarrow B'$ and graphs $E \rightrightarrows V$ and $E' \rightrightarrows V'$ yields the open transition system with span $A \times A' \leftarrow E \times E' \rightarrow B \times B'$ and graph $E \times E' \rightrightarrows V \times V'$.

3.2 The components of Span(Graph)

Let us now detail some useful constants of the algebra of **Span(Graph)**. We will illustrate how to use the algebra with an example in which we construct the NOR latch circuit from Figure 9.

Example 3. In this example, we model the circuit in Figure 9 in **Span(Graph)**_{*}. The connectivity of the circuit is modeled with a Frobenius algebra [10] (\blacktriangleleft , \rightarrow , \bullet , \bullet , \bullet). The corresponding spans are constructed out of diagonals $A \to A \times A$ and units $A \to 1$.

$$(-\!\!\!\!\!\bullet)_A = \{A \leftarrow A \rightarrow A \times A\} \qquad (-\!\!\!\!\!\bullet)_A = \{A \leftarrow A \rightarrow 1\}$$

$$(\mathbf{-})_A = \{A \times A \leftarrow A \rightarrow A\} \qquad (\mathbf{-})_A = \{1 \leftarrow A \rightarrow A\}$$

These already induce a compact closed structure (and, therefore, a trace), given by the following spans.

$$(\bullet - \bullet)_A = \{1 \leftarrow A \rightarrow A \times A\} \qquad (\triangleright \bullet)_A = \{A \times A \leftarrow A \rightarrow 1\}$$

In general, any function $f: A \to B$ can be lifted covariantly to a span $A \leftarrow A \to B$ and contravariantly to a span $A \leftarrow B \to B$. Any span $A \leftarrow E \to B$ can be lifted to $\mathbf{Span}(\mathbf{Graph})_*$ by making the head represent the graph $E \rightrightarrows 1$. We use this to obtain the graph of the NOR gate (Figure 2). However, components created like this have a single-vertex: they are stateless.

We will need a single stateful component to model our circuit, the delay

$$(-D-)_A = \left\{ \begin{array}{c} A \times A \\ \pi_2 & \pi_1 \\ A & A \end{array} \right\}.$$

This is *not* an arbitrary choice. This is the canonical delay obtained from the feedback structure⁷ in **Span**(**Graph**)** that gives its universal property.

The NOR latch circuit of Figure 9 is the composition of two NOR gates where the outputs of each gate has been copied and fed back as input to the other gate (Figure 17). The algebraic expression, in **Span(Graph)***, of this circuit is obtained by decomposing it into its components.

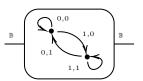


Fig. 16: Delay morphism over the set $\mathbb{B} := \{0, 1\}$.

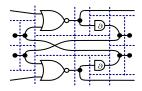


Fig. 17: Decomposing the circuit.

$$\begin{split} (\mathrm{id} \otimes \bullet \blacktriangleleft \otimes \bullet \blacktriangleleft \otimes \mathrm{id}); (\mathtt{NOR} \otimes \sigma \otimes \mathtt{NOR}); (\blacktriangleleft \otimes \mathrm{id} \otimes \blacktriangleleft) \\ & ; (\mathrm{id} \otimes \partial \otimes \mathrm{id} \otimes \partial \otimes \mathrm{id}); (\mathrm{id} \otimes \blacktriangleleft \bullet \otimes \blacktriangleleft \bullet \otimes \mathrm{id}) \end{split}$$

The graph obtained by the computation of this expression, together with its transitions, is shown in Figure 18. This time, our model is indeed stateful. It has four states: two states representing a correctly stored signal, $\overline{A} = (1,0)$ and A = (0,1); and two states representing transitory configurations $T_1 = (0,0)$ and $T_2 = (1,1)$.

We will be controlling the *left boundary*: it can receive a *set* signal, $\mathsf{Set} = \binom{1}{0}$; a reset signal, $\mathsf{Reset} = \binom{0}{1}$; none of the two, $\mathsf{Idle} = \binom{0}{0}$; or both of them at the same time, $\mathsf{Unspec} = \binom{1}{1}$, which is known to cause unspecified behavior in a NOR latch. The signal on the right boundary, on the other hand, is always equal to the state the transition goes to and does not provide any additional information. Knowing this, we omit it from the drawing in Figure 18.

⁷ As in Proposition 1.

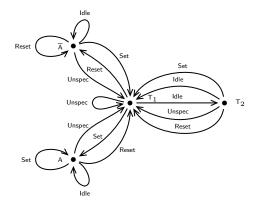


Fig. 18: Span of graphs representing the NOR latch

In normal functioning, activating the signal Set makes the latch transition to the state A in two transition steps. Analogously, activating Reset makes the latch transition to \overline{A} again in two transition steps. After any of these two cases, deactivating all signals, Idle, keeps the last state.

Moreover, the (real-world) NOR latch has some unspecified behavior that gets also reflected in the graph: activating both Set and Reset at the same time, what we call Unspec, causes the circuit to enter an unstable state where it bounces between the states T_1 and T_2 . Our modeling has reflected this "unspecified behavior" as expected.

Feedback and trace. In terms of feedback, the circuit of Figure 18 is equivalently obtained as the result of taking feedback over the following stateless morphism in Figure 19. We know that it is stateless because it is the composition of stateless morphisms.



Fig. 19: Applying fbk(•) over the circuit gives the NOR latch.

But **Span**(**Graph**)* is also canonically traced: it is actually compact closed. What changes in the modeling if, over the same morphism, we would have used trace instead? As we argued back for Figure 6, we obtain a stateless transition system: it is given by a graph with a single edge. The valid transitions can be now computed explicitly to be

$$\{(Unspec, T_1), (Idle, A), (Idle, \overline{A}), (Set, A), (Reset, \overline{A})\}$$

These encode important information: they are the *equilibrium* states of the circuit. However, unlike the previous graph, this one would not get us the correct allowed transitions: under this modeling, our circuit could freely bounce between (Idle, A) and $(Idle, \overline{A})$, which is not the expected behavior of a NOR latch.

The fundamental piece making our modeling succeed the previous time was feedback derived from the stateful *delay*. The next section explains in which sense that feedback is canonical.

3.3 Span(Graph) as a category with feedback

This section introduces our main theorem. We start by introducing the mapping that associates to each stateful span of sets the corresponding span of graphs. This mapping is well-defined and lifts to a functor $\mathsf{St}(\mathbf{Span}(\mathbf{Set})) \to \mathbf{Span}(\mathbf{Graph})$. Finally, we prove that it gives an equivalence $\mathsf{St}(\mathbf{Span}(\mathbf{Set})) \cong \mathbf{Span}(\mathbf{Graph})_*$.

Lemma 1. The following assignment of stateful processes over **Span(Set)** to morphisms of **Span(Graph)** is well defined.

$$K\left(S \middle| \begin{array}{c} E \\ (s,a) \middle \\ S \times A \\ S \times B \end{array}\right) := \left(\begin{array}{c} A \xleftarrow{a} E \xrightarrow{b} B \\ \left(\bigvee_{1} & s \middle(\bigvee_{1}^{t} & \bigvee_{1}^{t} \\ 1 \longleftarrow S \longrightarrow 1 \end{array}\right)$$

Proof. We first check that two *isomorphic* spans are sent to *isomorphic* spans of graphs. Let $S \times A \leftarrow E \rightarrow S \times B$ and $S \times A \leftarrow E' \rightarrow S \times B$ be two spans that are isomorphic with $h: E \cong E'$. Then (h, id) is an isomorphism of spans of graphs, also making the relevant diagram commute (Figure 20).

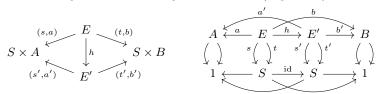


Fig. 20: Isomorphic spans result in isomorphic spans of graphs.

We show now that the assignment preserves the equivalence relation of stateful processes. Isomorphisms in a category of spans are precisely spans whose two legs are isomorphisms (Proposition 3). This means that an isomorphism in $\operatorname{Span}(\operatorname{Set})$ can be always rewritten as $S \leftarrow S \to T$, where the left leg is an identity and the right leg is $h \colon S \to T$, some isomorphism. Its inverse can be written analogously as $T \leftarrow S \to S$. In order to prove that the quotient relation induced by the feedback is preserved, we need to check that equivalent spans of sets are sent to isomorphic spans of graphs. If two spans are equivalent with the isomorphism $h \colon S \cong T$, then the corresponding graphs are isomorphic with the isomorphism of graphs (id, h).

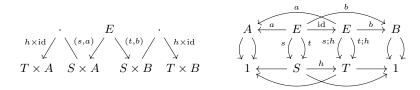


Fig. 21: Equivalent spans result in isomorphic spans of graphs.

Theorem 2. There exists an equivalence of categories

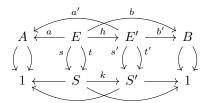
$$St(\mathbf{Span}(\mathbf{Set})) \cong \mathbf{Span}(\mathbf{Graph})_*$$
.

The free category with feedback over **Span**(**Set**) is equivalent to the full subcategory of **Span**(**Graph**) given by single-vertex graphs.

Proof. We prove that there is a fully faithful functor $K : \mathsf{St}(\mathbf{Span}(\mathbf{Set})) \to \mathbf{Span}(\mathbf{Graph})$ defined on objects as $K(A) = (A \rightrightarrows 1)$ and defined on morphisms as in Lemma 1. We have shown there that this assignation is well-defined.

We now show that it is functorial, preserving composition and identities. We can directly check that the identity morphism in $St(\mathbf{Span}(\mathbf{Set}))$, as a span $A \leftarrow A \rightarrow A$ is sent to the identity span of the graph $A \rightrightarrows 1$. Let us now show that composition is also preserved. The sequential composition of two stateful spans is computed as follows. Let the two stateful spans be given by $S \times A \leftarrow E \rightarrow S \times B$ and $T \times B \leftarrow F \rightarrow T \times C$, then the composite stateful span is given by $S \times T \times A \leftarrow E \times_B F \rightarrow S \times T \times C$. We check that this span is sent to the corresponding composition in $\mathbf{Span}(\mathbf{Graph})$. As \mathbf{Graph} is a functor category, limits are computed pointwise. Thus, the pullback of two graph morphisms is given by taking the pullbacks of both the vertices (where the pullback on 1 is a product) and the edges; this was shown in Figure 15.

The final step is to show that the original assignment is fully-faithful. We can see that it is full: every span of single-vertex graphs given by $A \leftarrow E \rightarrow B$ and $E \rightrightarrows S$ does arise from some span, namely $S \times A \leftarrow E \rightarrow S \times B$. Let us check it is also faithful. Suppose that two morphisms in $\mathsf{St}(\mathbf{Span}(\mathbf{Set})), S \times A \leftarrow E \rightarrow S \times B$ and $S' \times A \leftarrow E' \rightarrow S' \times B$, are sent to equivalent spans of graphs, i.e. there exist $h \colon E \cong E'$ and $k \colon S' \cong S$ making the following diagrams commute.



In this case, we know that $S \times A \leftarrow E \rightarrow S \times B$ is equivalent to $S' \times A \leftarrow E \rightarrow S' \times B$ because of the equivalence relation on stateful processes. Finally, $S' \times A \leftarrow E \rightarrow S' \times B$ is equivalent as a span to $S' \times A \leftarrow E' \rightarrow S' \times B$.

We have shown that there exists a fully-faithful functor from the free category with feedback over $\mathbf{Span}(\mathbf{Set})$ to the category $\mathbf{Span}(\mathbf{Graph})$ of spans of graphs. The functor induces an equivalence between $\mathsf{St}(\mathbf{Span}(\mathbf{Set}))$ and the full subcategory of $\mathbf{Span}(\mathbf{Graph})$ on single-vertex graphs.

3.4 Cospan(Graph) as a category with feedback

The previous results can be generalized to any category \mathbf{C} with all finite limits. By taking $\mathbf{Graph}(\mathbf{C})$ to be the presheaf category of the diagram $(\bullet \Rightarrow \bullet)$ in \mathbf{C} and $\mathbf{Span}(\mathbf{Graph}(\mathbf{C}))_*$ the full subcategory on objects of the form $A \Rightarrow 1$, we can prove the following result.

Theorem 3. There exists an equivalence of categories

$$St(\mathbf{Span}(\mathbf{C})) \cong \mathbf{Span}(\mathbf{Graph}(\mathbf{C})).$$

The free category with feedback over $\mathbf{Span}(\mathbf{C})$ is equivalent to the full subcategory on $\mathbf{Span}(\mathbf{Graph}(\mathbf{C}))$ given by single-vertex graphs.

 $\mathbf{Cospan}(\mathbf{Graph})_*$ can be also characterized as a free category with feedback. We know that $\mathbf{Cospan}(\mathbf{Set}) \cong \mathbf{Span}(\mathbf{Set}^{op})$, we note that $\mathbf{Graph}(\mathbf{Set}^{op}) \cong \mathbf{Graph}$ (which has the effect of flipping edges and vertices), and we can use Theorem 3 because \mathbf{Set} has all finite colimits. The explicit assignment is similar to the one shown in Lemma 1.

$$K\left(S \middle| \begin{array}{c} S \\ [t|a] \nearrow \\ E+A \end{array} \right) := \left(\begin{matrix} A \stackrel{a}{\longrightarrow} S \stackrel{b}{\longleftarrow} B \\ \nearrow \\ 0 \longrightarrow E \longleftarrow 0 \end{matrix}\right)$$

Corollary 1. There exists an equivalence of categories $St(Cospan(Set)) \cong Cospan(Graph)_*$.

Two feedback structures on Cospan(Graph). Cospan(Graph) is also compact closed and, in particular, traced. As in the case of Span(Graph), the feedback structure given by the universal property is different from the trace. While the trace identifies the vertices that are the images of the same element on the boundaries, the feedback puts an additional edge between them.

3.5 Syntactical presentation of Cospan(FinGraph)

The observation in Proposition 1 has an important consequence in the case of finite sets. Let us call **FinGraph** to **Graph**(**FinSet**). **Cospan**(**FinSet**) is the generic special commutative Frobenius algebra [30], meaning that any morphism written out of the operations of a special commutative Frobenius algebra and the structure of a symmetric monoidal category is precisely a cospan of finite sets (or, in other words, symmetric monoidal functors out of **Cospan**(**FinSet**)

correspond to special commutative Frobenius algebras). But we also know that $\mathbf{Cospan}(\mathbf{FinSet})$ with an added generator to its PROP structure [7] (the delay, with the conditions given in Proposition 1) is $\mathsf{St}(\mathbf{Cospan}(\mathbf{FinSet}))$, or, equivalently, $\mathbf{Cospan}(\mathbf{FinGraph})$. This means that any morphism written out of the operations of a special commutative Frobenius algebra plus a freely added generator of type (-): $1 \to 1$ is a morphism in $\mathbf{Cospan}(\mathbf{FinGraph})_*$. This is a direct proof of a fact that already appeared in [35].

Proposition 2 ([35], Proposition 3.2). The category Cospan(FinGraph)_{*} is the generic special commutative Frobenius monoid with an added generator.

Proof. It is known that the category **Cospan**(**FinSet**) is the generic special commutative Frobenius algebra [30]. Adding a free generator ($\neg \bigcirc$): $1 \to 1$ to its PROP structure corresponds to adding a family $(\neg \bigcirc -)_n : n \to n$ with the conditions on Proposition 1. Now, Proposition 1 implies that adding such a generator to **Cospan**(**FinSet**) results in **St**(**Cospan**(**FinSet**)). Finally, we can use again Theorem 2 to conclude that **St**(**Cospan**(**FinSet**)) \cong **Cospan**(**FinGraph**)*.

4 Conclusions and further work

We have characterized $\mathbf{Span}(\mathbf{Graph})_*$, an algebra of open transition systems, as equivalent to the free category with feedback over the category of spans of functions. The $\mathsf{St}(\bullet)$ constuction is well-known as a technique of adding state to processes. In [28], it had been characterized as the free category with feedback under a different name. What was missing was a coherent and explicit connection between the two.

We have seen how the $\mathsf{St}(\bullet)$ construction creates categories of transition systems out of symmetric monoidal categories. As future work, we plan to study how to expand our investigation from mere transition systems to automata, which have initial and accepting states. Initial states are particularly important when providing semantics: for instance, we would like to interpret a transition system $S \times A \to S \times B$ as a stream transducer $\mathbf{Stream}(A) \to \mathbf{Stream}(B)$, but this is not possible without an initial state $s_0 \in S$. In future work, we discuss an elegant way of accomodating initial states by considering a more general definition of feedback. This generalized definition modifies the sliding axiom: instead of sliding isomorphisms, we can slide arbitrary classes of morphisms under the image of a functor. We conjecture that sliding point-preserving homomorphisms of pointed sets recovers a well-behaved notion of initial state without modifying the framework of categories with feedback any further.

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Appendix

Remarks

Remark 1. An alternative definition of feedback, $\mathsf{fbk}(\bullet)$, declares it to be an operator taking instead a morphism $S \otimes A \to B \otimes S$, yielding a morphism $A \to B$.

$$\frac{f \colon S \otimes A \to B \otimes S}{\mathsf{fbk}_S(f) \colon A \to B}$$

Borrowing the names from the exposition of traces in [34], we will call this twisted feedback, and contrast it with the aligned feedback we have decided to define instead. Let us explain the rationale behind this decision. The advantage of using twisted feedback is that we can define sequential composition of stateful processes without ever requiring symmetry of the underlying monoidal category (as in [24]). However, the parallel composition does require symmetry in any case; and diagrams seem easier to draw and read when using aligned feedback.

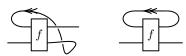


Fig. 22: Twisted vs. aligned feedback

Omitted definitions: Monoidal categories

Definition 8. A monoidal category $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ is a category \mathbf{C} equipped with a functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, a unit $I \in \mathbf{C}$, and three natural isomorphisms: the associator $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, the left unitor $\lambda_A : I \otimes A \cong A$ and the right unitor $\rho_A : A \otimes I \cong A$; such that $\alpha_{A,I,B} : (\mathrm{id}_A \otimes \lambda_B) = \rho_A \otimes \mathrm{id}_B$ and $(\alpha_{A,B,C} \otimes \mathrm{id}) : \alpha_{A,B\otimes C,D} : (\mathrm{id}_A \otimes \alpha_{B,C,D}) = \alpha_{A\otimes B,C,D} : \alpha_{A,B,C\otimes D}$. A monoidal category is strict if α , λ and ρ are identities.

Definition 9. Let $(\mathbf{C}, \otimes, I, \alpha^{\mathbf{C}}, \lambda^{\mathbf{C}}, \rho^{\mathbf{C}})$ and $(\mathbf{D}, \boxtimes, J, \alpha^{\mathbf{D}}, \lambda^{\mathbf{D}}, \rho^{\mathbf{D}})$ be monoidal categories. A monoidal functor (or strong monoidal functor) is a triple (F, ε, μ) consisting of a functor $F: \mathbf{C} \to \mathbf{D}$ and two natural isomorphisms $\varepsilon: J \cong F(I)$ and $\mu: F(A \otimes B) \cong F(A) \boxtimes F(B)$; such that the associators satisfy $\alpha^{\mathbf{D}}_{FA,FB,FC}$; $(\mathrm{id}_{FA} \otimes \mu_{B,C})$; $\mu_{A,B \otimes C} = (\mu_{A,B} \otimes \mathrm{id}_{FC})$; $\mu_{A \otimes B,C}$; $F(\alpha^{\mathbf{C}}_{A,B,C})$, the left unitor satisfies $(\varepsilon \otimes \mathrm{id}_{FA})$; $\mu_{I,A}$; $F(\lambda^{\mathbf{C}}_A) = \lambda^{\mathbf{D}}_{FA}$ and the right unitor satisfies $(\mathrm{id}_{FA} \otimes \varepsilon)$; $\mu_{A,I}$; $F(\rho^{\mathbf{C}}_{FA}) = \rho^{\mathbf{D}}_{FA}$. A monoidal functor is a monoidal equivalence if it is moreover an equivalence of categories. Two monoidal categories are monoidally equivalent if there exists a monoidal equivalence between them.

Theorem 4 (Coherence theorem, [31]). Every monoidal category is monoidally equivalent to a strict monoidal category.

Let us comment further on how we use the coherence theorem. Each time we have a morphism $f: A \to B$ in a monoidal category, we have a corresponding

morphism $A \to B$ in its strictification. This morphism can be lifted to the original category to uniquely produce, say, a morphism $(\lambda_A; f; \lambda_B^{-1}): I \otimes A \to I \otimes B$. Each time the source and the target are clearly determined, we simply write f again for this new morphism.

As an example, consider the statement of the vanishing axiom, which says that for $f \colon A \to B$, we have $\mathsf{fbk}_I(f) = f$. However, the feedback operator needs to be applied to a morphism $I \otimes A \to I \otimes B$, and the only such morphism that is mapped again to $f \colon A \to B$ in the equivalent strict monoidal category is $(\lambda_A; f; \lambda_B^{-1}) \colon I \otimes A \to I \otimes B$. Thus, we are really stating that

$$\mathsf{fbk}_I(\lambda_A; f; \lambda_B^{-1}) = f.$$

In the same vein, the joining axiom really states that

$$\mathsf{fbk}_S(\mathsf{fbk}_T(f)) = \mathsf{fbk}_{S \otimes T}(\alpha_{S,T,A}; f; \alpha_{S,T,B}^{-1}).$$

The reason to avoid this explicit notation on our definitions and proofs is that it would quickly become verbose and distractive. Equations seem conceptually easier to understand when written assuming the coherence theorem. And we are following this criterion anyway when employing string diagrams. In fact, in [28], strictness is assumed since the beginning to facilitate the reading, even when we have shown it is not a necessary assumption.

There are cases where we do need to be careful about the correct use of associators and unitors. For instance, when writing stateful processes, we could be tempted to conclude that, for any $f: ((S \otimes T) \otimes R) \otimes A \to ((S \otimes T) \otimes R) \otimes B$, the following equation holds $((S \otimes T) \otimes R \mid f) = (S \otimes (T \otimes R) \mid f)$ without needing to invoke the equivalence relation. This would allow us to construct the category $\mathsf{St}(\bullet)$ of stateful processes without having to quotient them by the equivalence relation. However, this equality is only enabled by the fact that $\alpha_{S,T,R}$ is an isomorphism: we have

$$((S \otimes T) \otimes R \mid f) = (S \otimes (T \otimes R) \mid \alpha_{S,R,T}; f; \alpha_{S,R,T}^{-1}),$$

even if we write the equation omitting the coherence maps. This is also what allows us to notate stateful processes diagramatically. We only mark the wires forming the space state; the order in which they are tensored does not matter thanks again to the equivalence relation that we are imposing.

Definition 10. A symmetric monoidal category $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is a monoidal category $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ equipped with a braiding $\sigma_{A,B} \colon A \otimes B \to B \otimes A$, which satisfies the hexagon equation $\alpha_{A,B,C}$; $\sigma_{A,B\otimes C}$; $\alpha_{B,C,A} = (\sigma_{A,B}\otimes \mathrm{id})$; $\alpha_{B,A,C}$; $(\mathrm{id}\otimes \sigma_{A,C})$ and additionally satisfies $\sigma_{A,B}$; $\sigma_{B,A} = \mathrm{id}$.

Definition 11. A symmetric monoidal functor $F: \mathbf{C} \to \mathbf{D}$ between two symmetric monoidal categories $(\mathbf{C}, \sigma^{\mathbf{C}})$ and $(\mathbf{D}, \sigma^{\mathbf{D}})$ is a strong monoidal functor such that $\sigma^{\mathbf{D}}$; $\psi = \psi$; $F(\sigma^{\mathbf{C}})$.

Definition 12. A traced monoidal category [22,37] is a category with feedback that additionally satisfies the yanking axiom $\mathsf{fbk}(\sigma) = \mathsf{id}$ and that additionally satisfies the sliding axiom, $\mathsf{fbk}_T(f;(h\otimes \mathsf{id})) = \mathsf{fbk}_S((h\otimes \mathsf{id});f)$, for an arbitrary morphism $h\colon S\to T$.

Omitted proofs: spans

Proposition 3. Let \mathbb{C} be a category with all finite limits. In its category of spans, $\mathbf{Span}(\mathbb{C})$, an isomorphism $A \cong B$ is always of the form $A \leftarrow A \rightarrow B$, where the left leg is an identity and the right leg is an isomorphism in \mathbb{C} .

Proof. Let $A \leftarrow E \rightarrow B$ and $B \leftarrow E' \rightarrow A$ be inverses. This means that the following pullback diagrams commute.

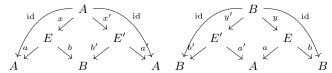


Fig. 23: Pullback diagrams representing compositions in **Span**(**C**).

We know then that a, a', b and b' are split epimorphisms, whereas x, x', y and y' are their corresponding split monomorphisms. Let us prove that they are also isomorphisms.

The span $E \leftarrow E \rightarrow E'$, given by $b; y' \colon E \rightarrow E'$ and the identity, is a cone over the diagram $E \rightarrow B \leftarrow E'$, because $y'; b' = \mathrm{id}_B$. By the universal property of the pullback, there exists a unique $h \colon E \rightarrow A$ such that $h; x = \mathrm{id}_E$ and h; x' = b; y'. This proves that x is a split epimorphism and, hence, an isomorphism. The same reasoning can be repeated for x', y and y'. It follows that a, b, a', b' are isomorphisms as well.

Once we have shown that the original $A \leftarrow E \rightarrow B$ and $B \leftarrow E' \rightarrow A$ have pairs of isomorphisms as legs, we can rewrite them as

$$A \stackrel{\text{id}}{\longleftarrow} A \stackrel{a^{-1};b}{\Longrightarrow} B$$
 and $A \stackrel{a'^{-1};b'}{\longleftarrow} B \stackrel{\text{id}}{\longrightarrow} B$,

where one leg is an identity and the other is an isomorphism.