Open Diagrams via Coend Calculus

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Applied Category Theory 2020, MIT

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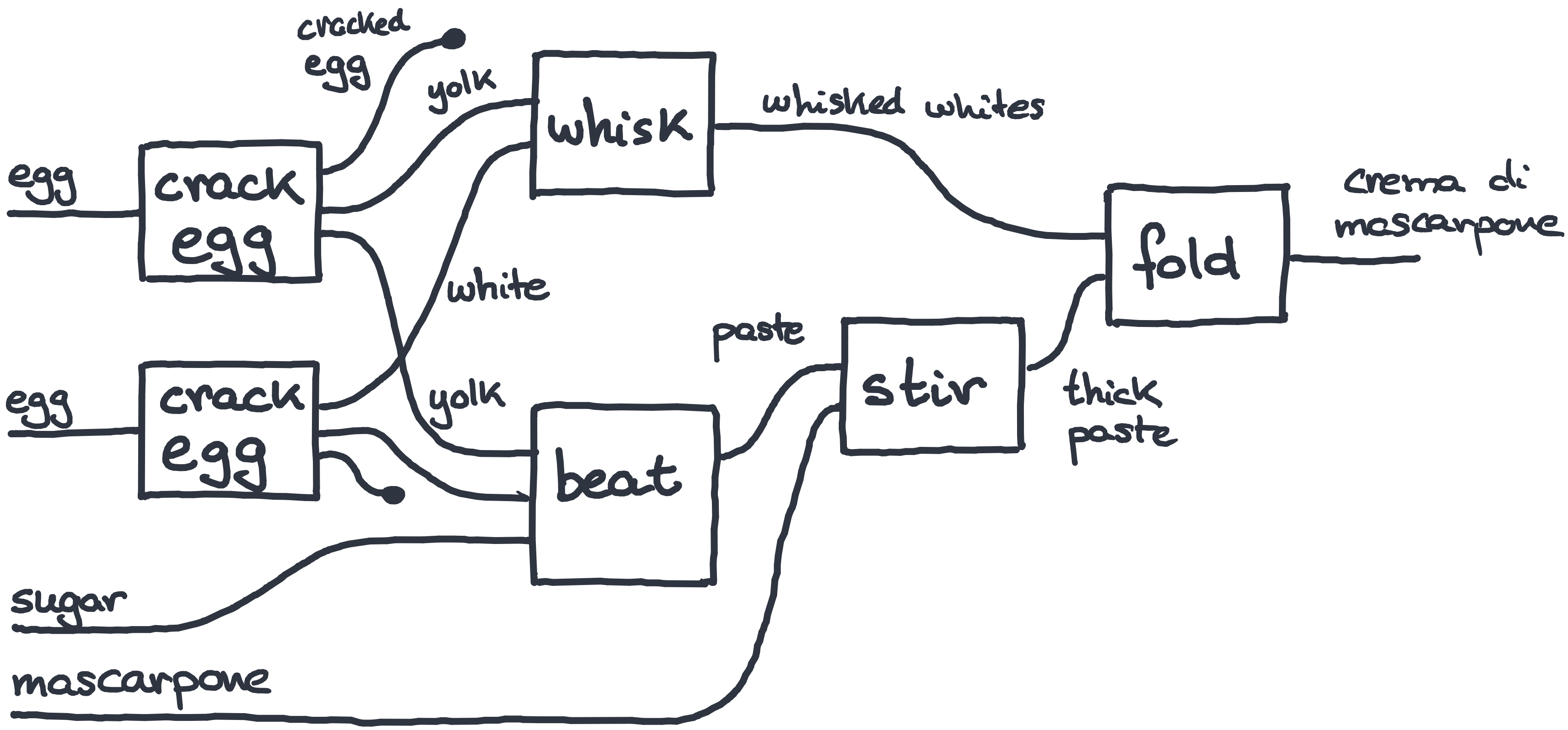
Part 1: Open Diagrams

... or "Incomplete String Diagrams"

... or "Diagrams with holes"

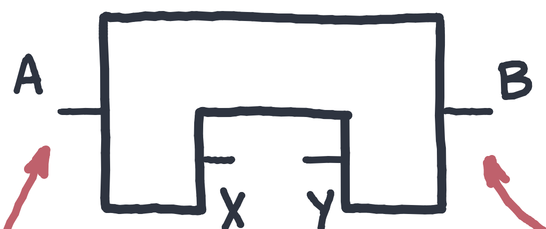
String Diagrams

- Process interpretation.



String Diagrams

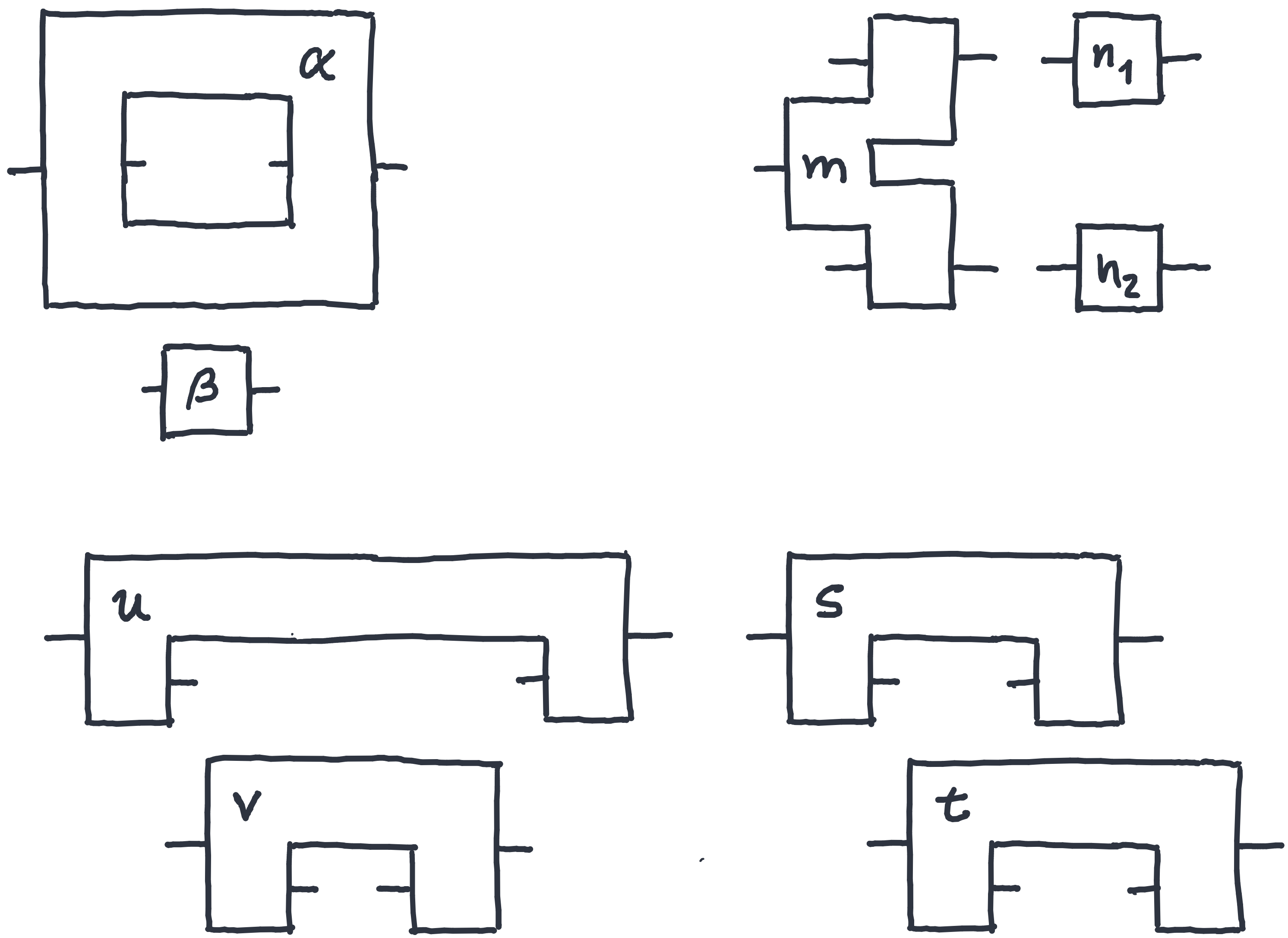
- How to interpret processes in multiple stages?



Take input A, produce output X, only then, take input Y, output B.

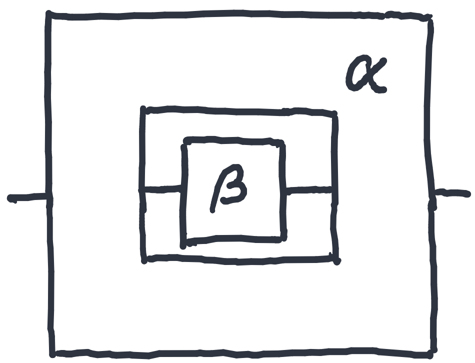
String Diagrams

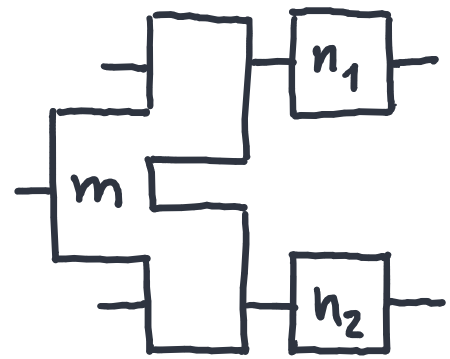
- How to compose processes in multiple stages?

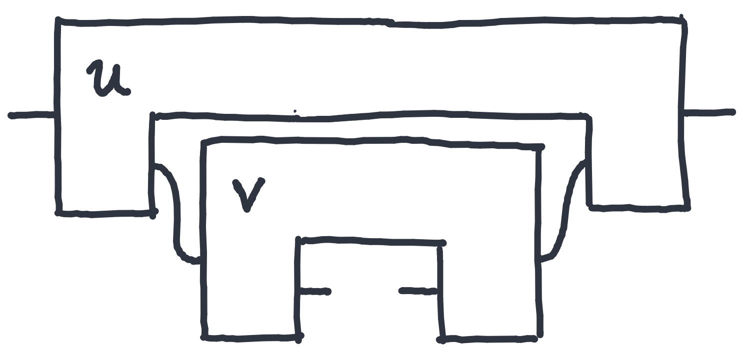


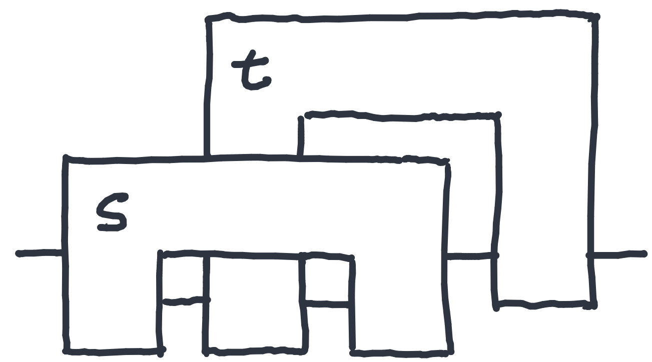
String Diagrams

- How to compose processes in multiple stages?



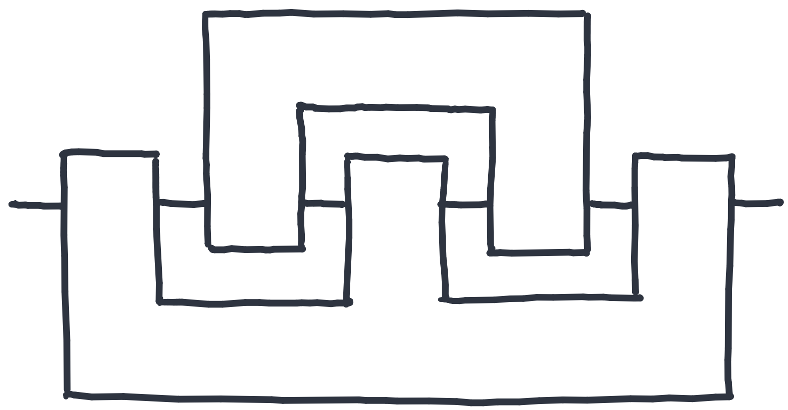




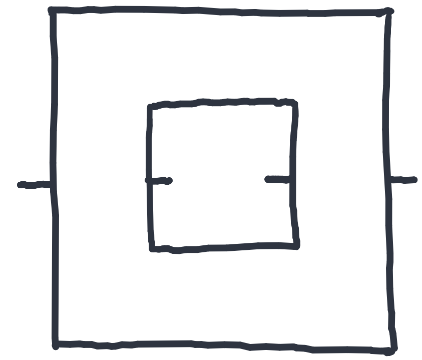


String Diagrams

- Quotiented tuples.

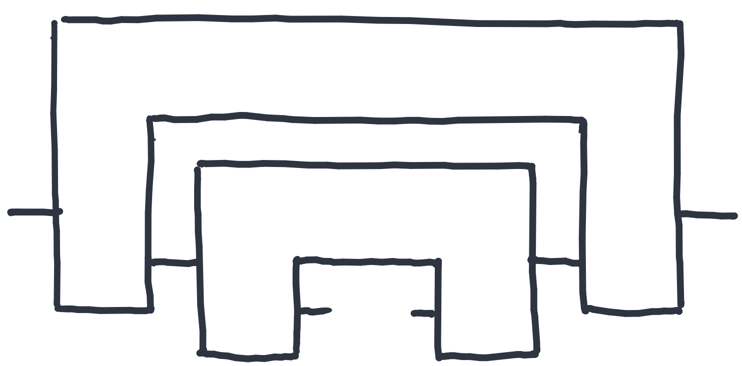


J. Hedges, Open Games

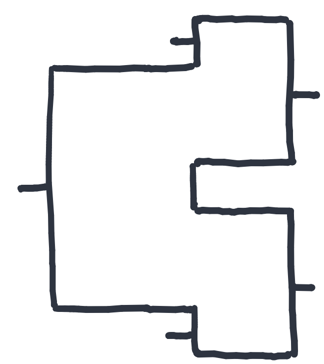


Chiribella, D'Ariano, Perinotti

Quantum Circuit Architecture.



M. Riley, Categories of optics



N. Virgo, IO machines.

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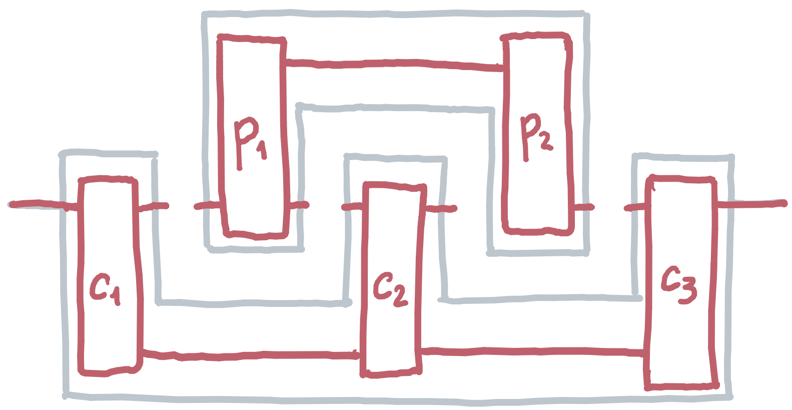

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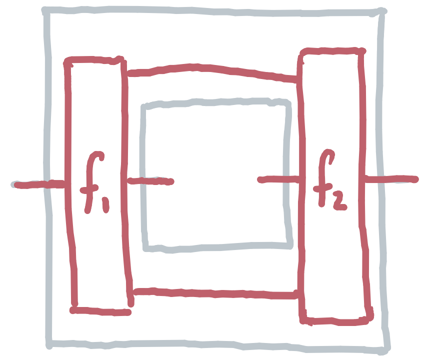

Chiribella, D'Ariano, Perinotti

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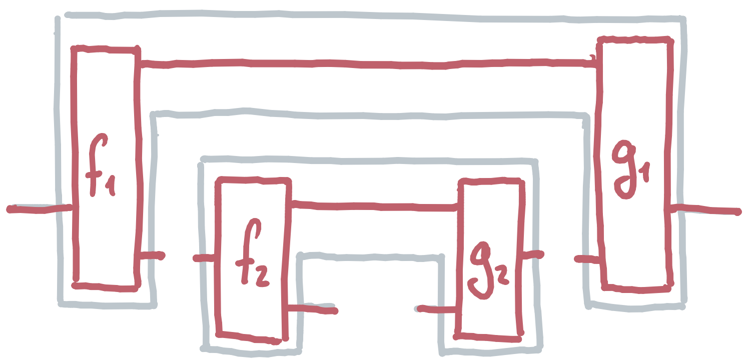

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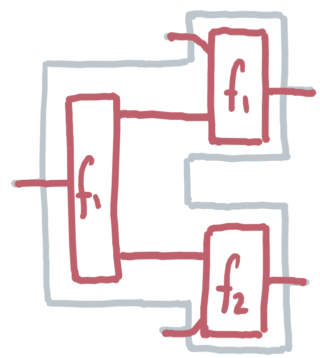

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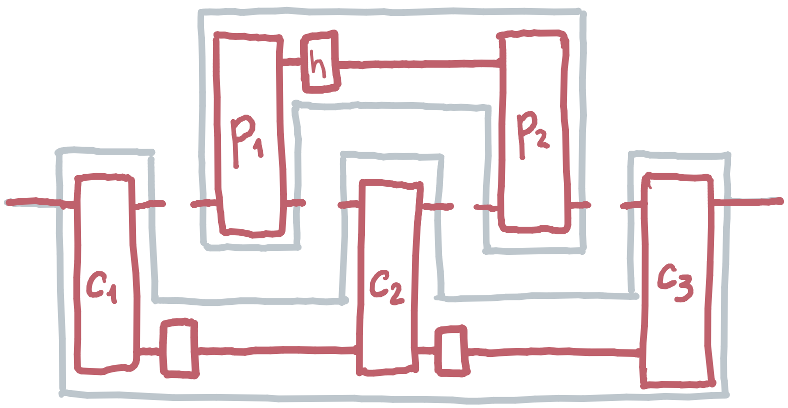

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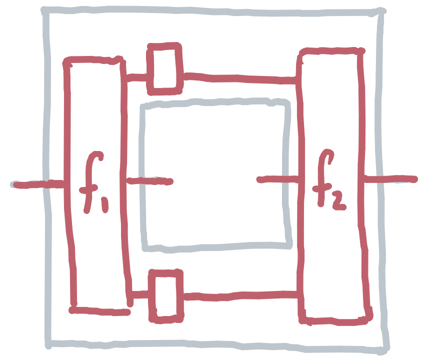

Chiribella, D'Ariano, Perinotti

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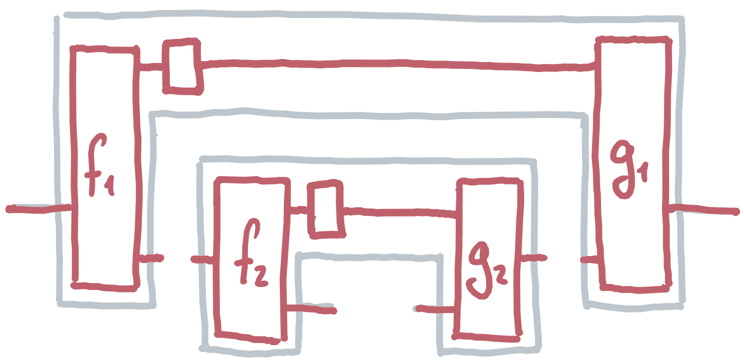

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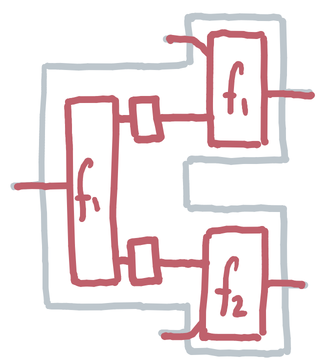

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String Diagrams

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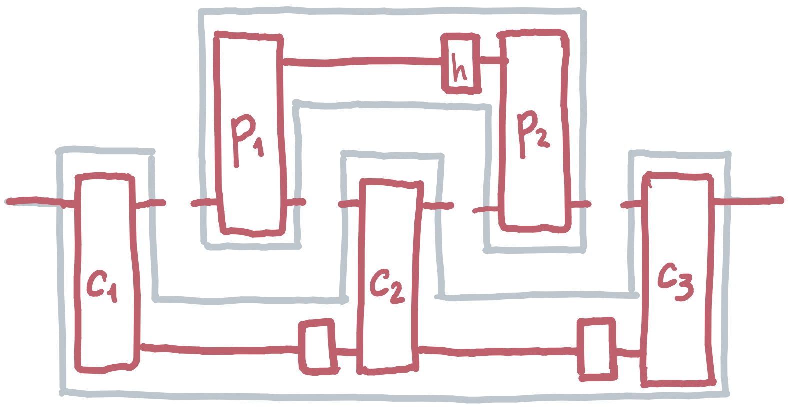
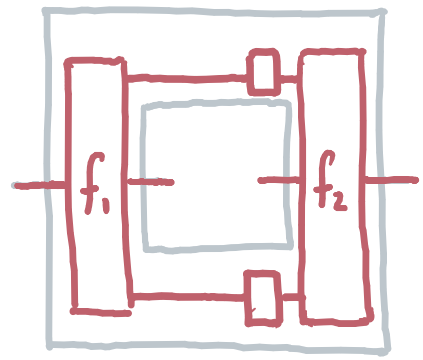
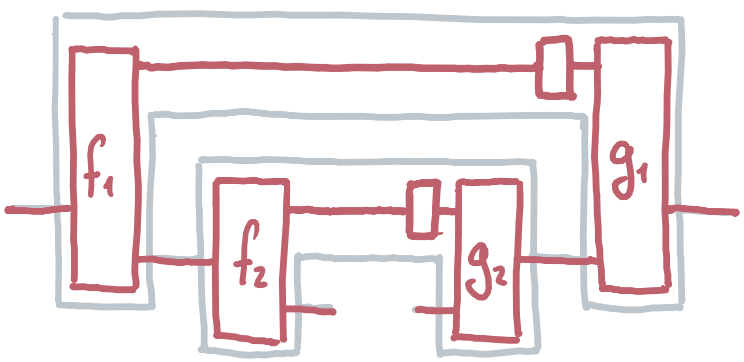


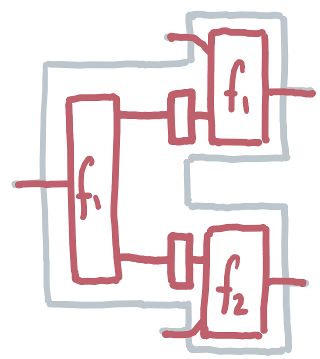
J. Hedges, Open Games



Chiribella, D'Ariano, Perinotti
Quantum Circuit Architecture.

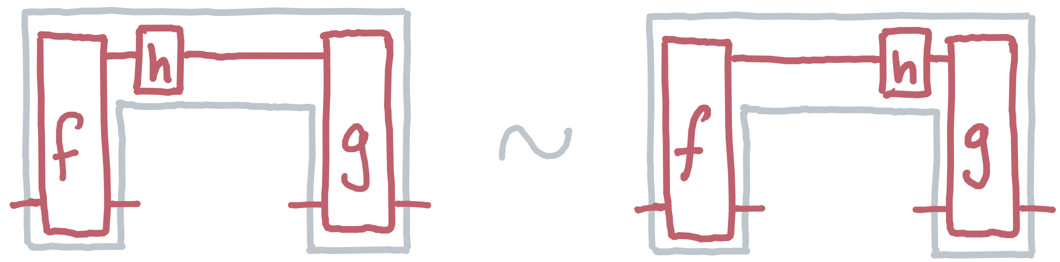


M. Riley, Categories of optics



N. Virgo, IO machines.

This is the coend



$$\langle f \fatsemi (h \otimes id) \mid g \rangle \sim \langle f \mid (h \otimes id) \fatsemi g \rangle$$

- The quotienting coincides with that of a coend.

$$\int^{M \in \mathbf{A}} \mathbf{A}(A, M \otimes X) \times \mathbf{A}(M \otimes Y, B)$$

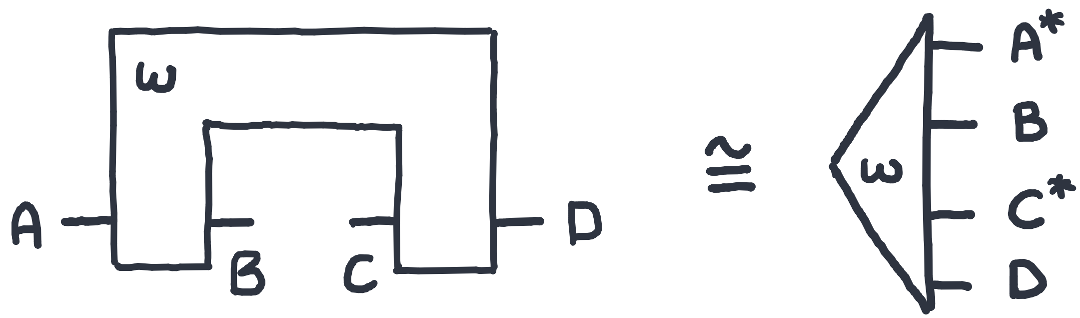
"Elements of Optic(...) have an appealing interpretation as string diagrams with a hole missing." - Riley, 2017.

This is the coend

- Advantage: we can compute.

- **Proposition.** In a compact closed category $\mathbb{A}$,

$$\int^{M\in\mathbb{A}} \mathbb{A}(A, M\otimes X)\times\mathbb{A}(M\otimes Y, B) \cong \mathbb{A}(I, A^*\otimes B\otimes C^*\otimes D).$$



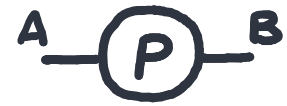
"Using 'comb' notation, (...) we allow for irregularly-shaped boxes."
- Kissinger, Uijlen 2017

Part 2: Profunctors

Monoidal bicategory of profunctors.

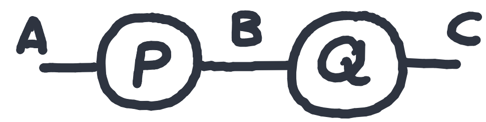
- Small categories and profunctors.

$$P : \mathcal{A}^{op} \times \mathcal{B} \to \mathbf{Set} \qquad \text{or} \qquad P : \mathcal{A} \nrightarrow \mathcal{B}$$



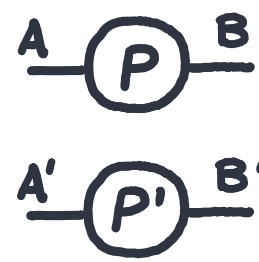
- Composition by coends.

$$(P \diamond Q)(A,C) = \int^{B \in \mathcal{B}} P(A,B) \times Q(B,C)$$



- Monoidal product by the cartesian product.

$$(P \otimes Q)(A,A',B,B') = P(A,B) \times Q(A',B')$$



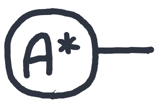
- 2-cells are natural transformations.

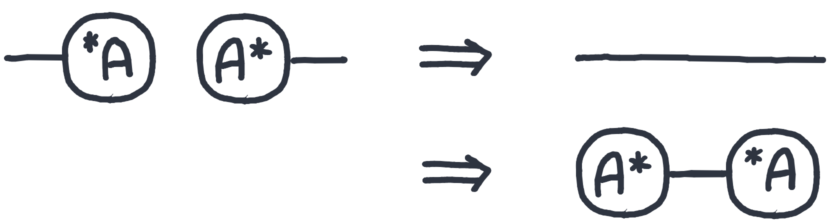
Bartlett, Quasistrict Symmetric Monoidal 2-Categories via wire diagrams.

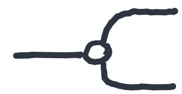
Monoidal bicategory of profunctors.

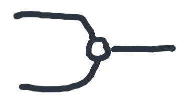
- The Yoneda embeddings $\mathbf{Cat} \to \mathbf{Prof}$ determine adjoints.

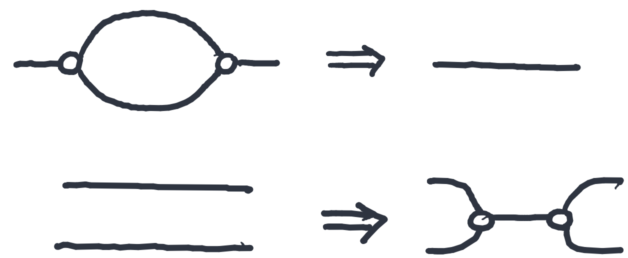
 : $A \nrightarrow 1$

 : $1 \nrightarrow A$



 : $A \nrightarrow A \times A$

 : $A \times A \nrightarrow A$



- Not completely strict.

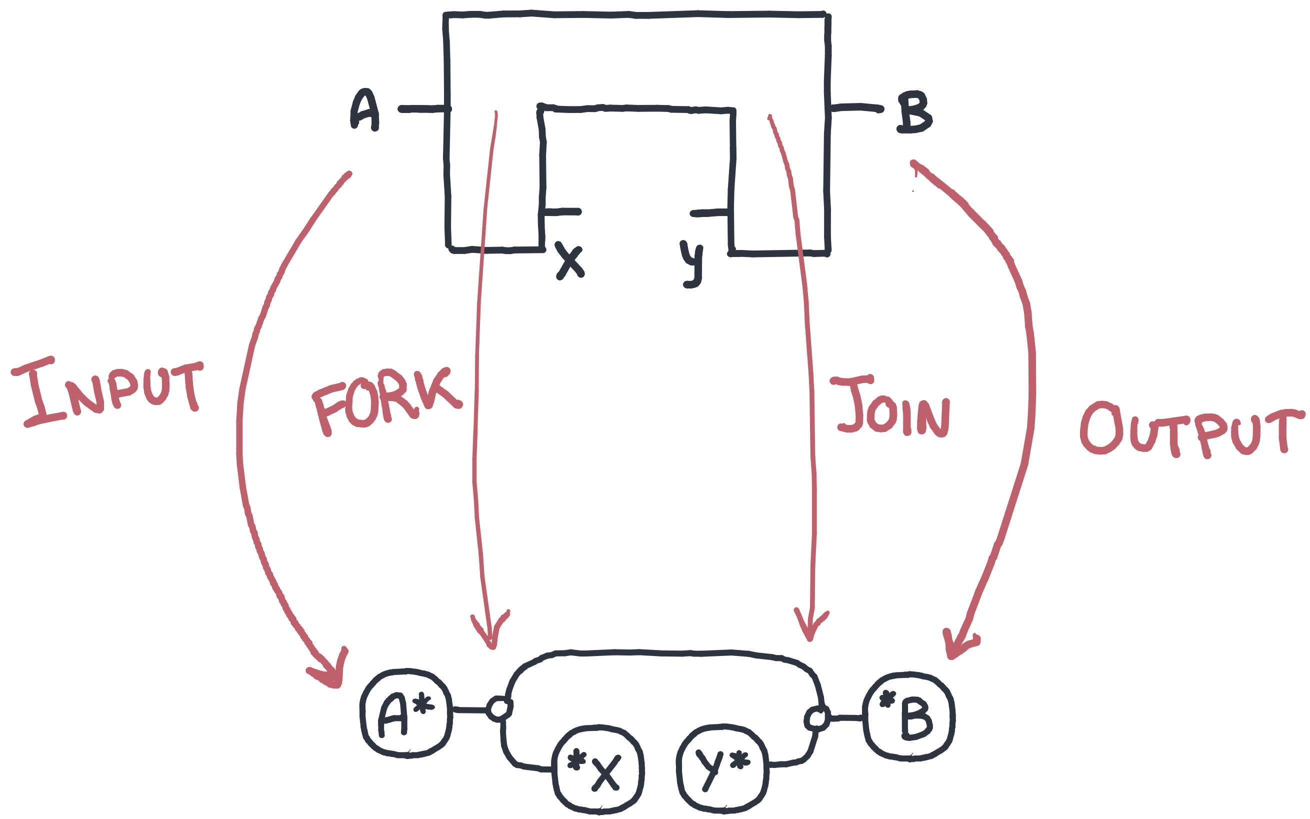
"Interchangor"

- Diagrams without input/output wires are sets. $1 \nrightarrow 1$ means $1^{op} \times 1 \longrightarrow \mathbf{Set}$.

Bartlett, Quasistrict Symmetric Monoidal 2-Categories via wire diagrams.

Monoidal bicategory of profunctors.

- We can calculate shapes.



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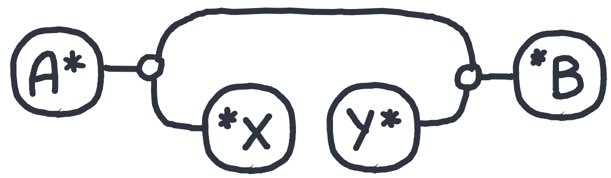

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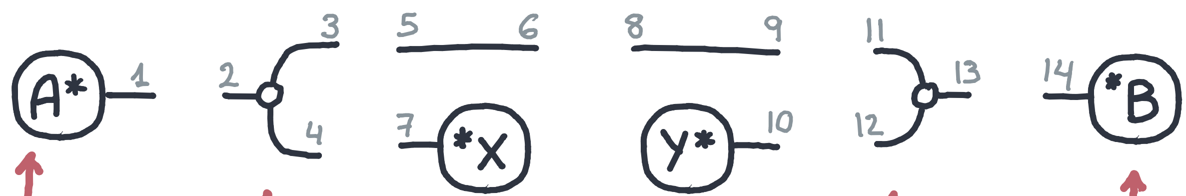

$\hom(A, \overset{1}{-})$

$\hom(\overset{2}{-}, \overset{3}{-} \otimes \overset{4}{-})$

$\hom(\overset{5}{-}, \overset{6}{-}) \times \hom(\overset{7}{-}, X)$

$\hom(\overset{8}{-}, \overset{9}{-}) \times \hom(Y, \overset{10}{-})$

$\hom(\overset{11}{-} \otimes \overset{12}{-}, \overset{13}{-})$

$\hom(\overset{14}{-}, B)$

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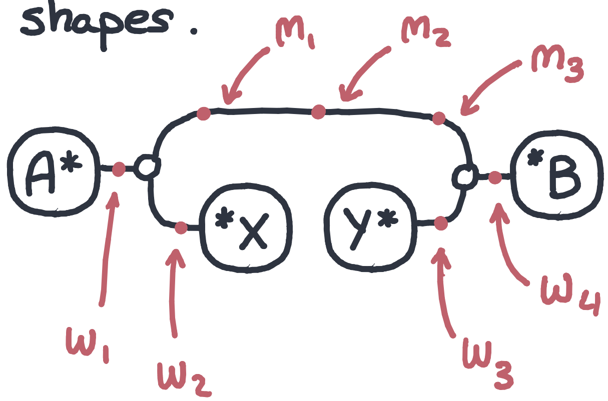
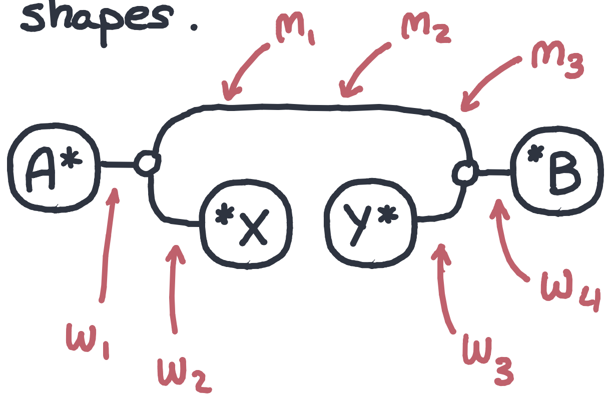


$$\int^{w_1,w_2,w_3,w_4,m_1,m_2,m_3} \hom(A,w_1)\times \hom(w_1, w_2\otimes m_1)\times \hom(m_1,m_2)\times \hom(w_2,X)\times \hom(m_2,m_3)\times$$
$$\hom(Y,w_3)\times \hom(m_3\otimes w_3, w_4)\times \hom(w_4,B)$$

Monoidal bicategory of profunctors.

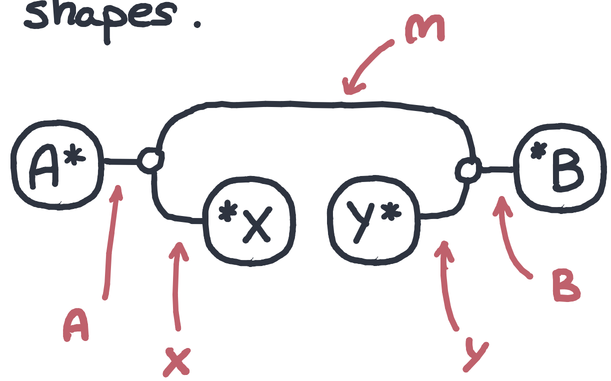
- We can calculate shapes.



$$\int^{w_1, w_2, w_3, w_4, m_1, m_2, m_3} \hom(A, w_1) \times \hom(w_1, w_2 \otimes m_1) \times \hom(m_1, m_2) \times \hom(w_2, X) \times \hom(m_2, m_3) \times \hom(Y, w_3) \times \hom(m_3 \otimes w_3, w_4) \times \hom(w_4, B)$$

Monoidal bicategory of profunctors.

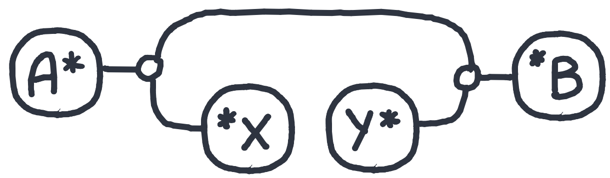
- We can calculate shapes.



$$\int^{\overset{A}{w_1},\,\overset{X}{w_2},\,\overset{Y}{w_3},\,\overset{B}{w_4},\,\overset{M}{m_1},\,\overset{M}{m_2},\,\overset{M}{m_3}} \cancel{\hom(A,w_1)} \times \hom(w_1, w_2\otimes m_1) \times \cancel{\hom(m_1,m_2)} \times \cancel{\hom(w_2,X)} \times \cancel{\hom(m_2,m_3)} \times \cancel{\hom(Y,w_3)} \times \hom(m_3\otimes w_3, w_4) \times \cancel{\hom(w_4,B)}$$

Monoidal bicategory of profunctors.

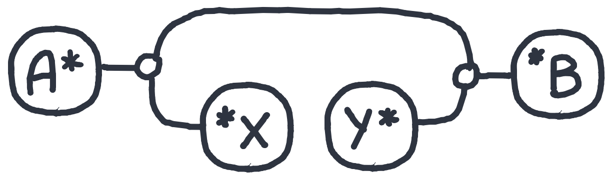
- We can calculate shapes.



$$\int^{M} \hom(A, X \otimes M) \times \hom(M \otimes Y, B)$$

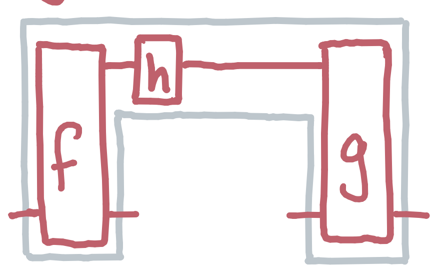
Monoidal bicategory of profunctors.

- We can calculate shapes.



$$\int^{M} \hom(A, X\otimes M) \times \hom(M\otimes Y, B)$$

- How to recover the diagrams we were drawing in a monoidal bicategory?



Monoidal bicategory of pointed profunctors.

- Small categories with a chosen object.

  $(\mathcal{A}, A)$; $(\mathcal{B}, B)$; $(\mathcal{A}', A)$

- Profunctors, carrying a point.

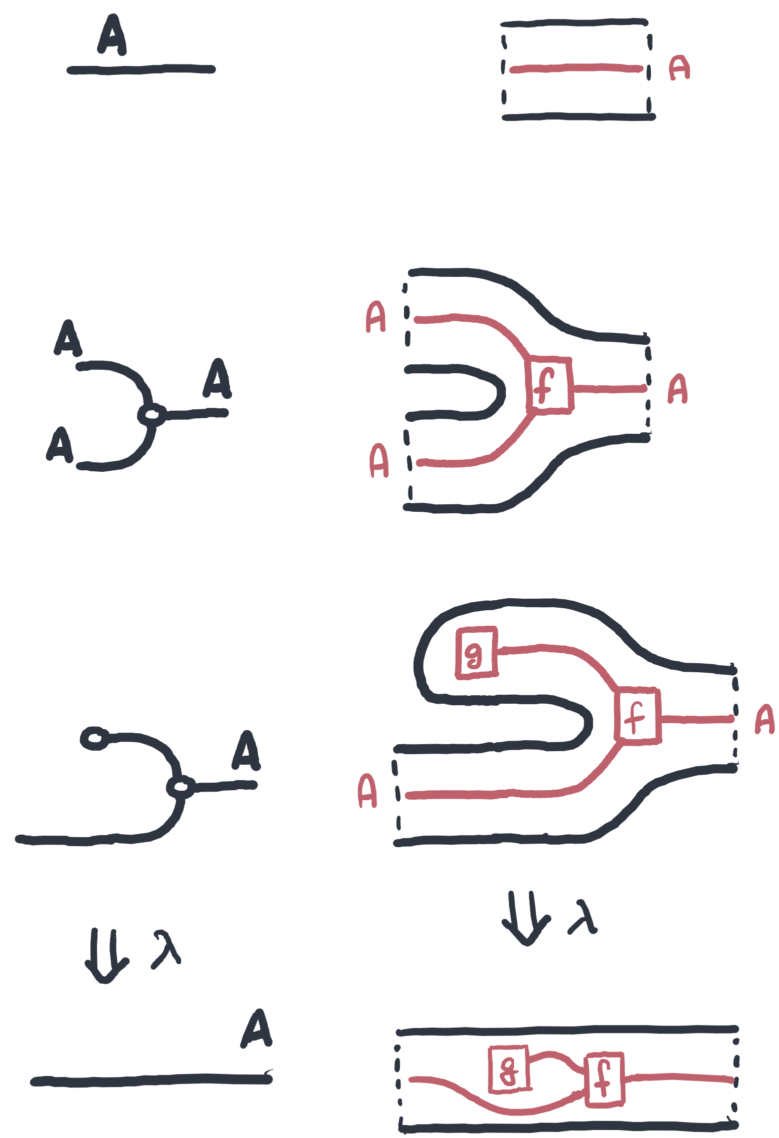
  $(p, P) : (\mathcal{A}, A) \longrightarrow (\mathcal{B}, B)$

  where

  $P : \mathcal{A}^{op} \times \mathcal{B} \to \mathbf{Set}$

  $p \in P(A, B)$

- Natural transformations preserve the point.



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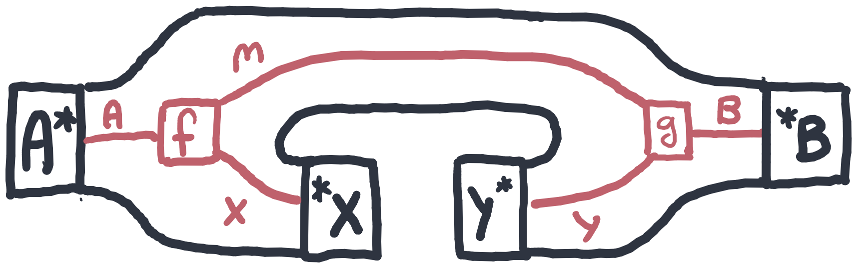

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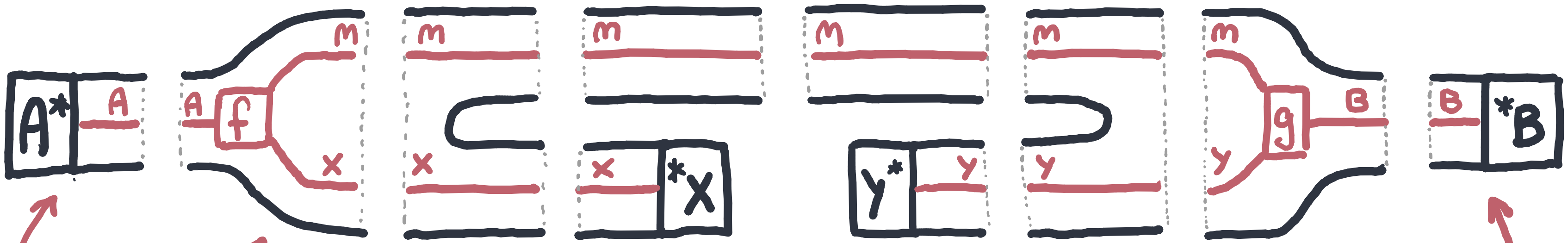


$id \in hom(A, A)$

$f \in hom(A, X \otimes M)$

$id \in hom(B \otimes C, B \otimes C)$

$(id, id) \in hom(M, M) \times hom(X, X)$

$(id, id) \in hom(M, M) \times hom(Y, Y)$

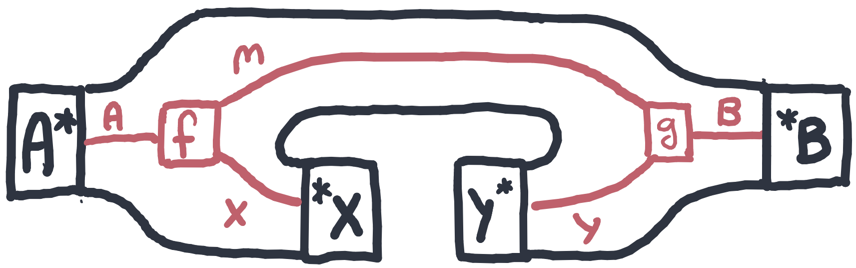
$id \in hom(M \otimes Y, M \otimes Y)$

$g \in hom(M \otimes Y, B)$

$id \in hom(B, B)$

Lenses

- We can calculate shapes.

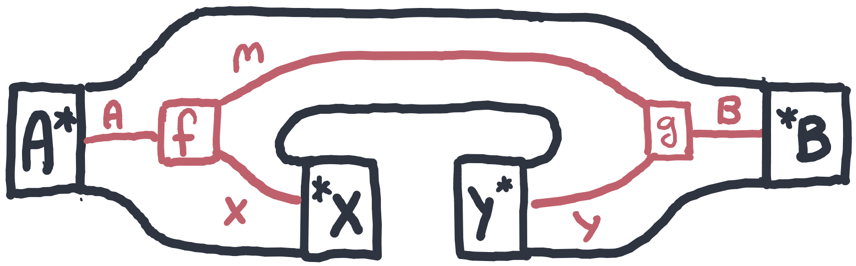


$$\langle id, f, id, (id, id), (id, id), id, g, id \rangle \in$$

$$\int^{H_1, H_2 \ldots} hom(A, A) \times hom(A, X \otimes M) \times hom(B \otimes C, B \otimes C) \times hom(M, M) \times hom(X, X)$$
$$hom(M, M) \times hom(Y, Y) \times hom(M \otimes Y, M \otimes Y) \times hom(M \otimes Y, B) \times hom(B, B)$$

Lenses

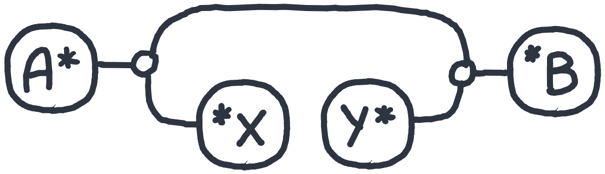
- We can calculate shapes.



$$\langle f, g \rangle \in \int^{M \in A} \hom(A, M \otimes X) \times \hom(M \otimes Y, B)$$

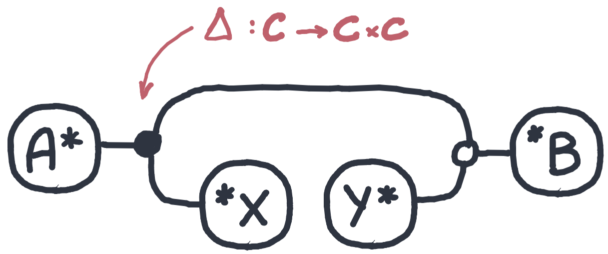
Part 3: Examples

Coend derivations



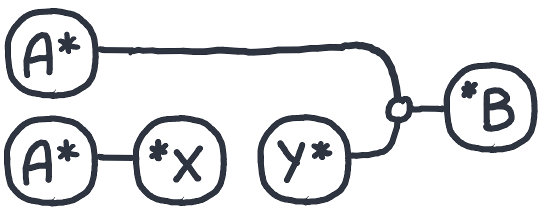
$$\int^{M} \hom_{C}(A, X \star M) \times \hom_{C}(M \star Y, B)$$

$$\cong$$



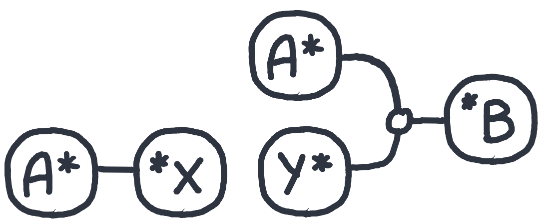
$$\int^{M} \hom_{C\times C}(\Delta(A), X \star M) \times \hom_{C}(M \star Y, B)$$

$$\cong$$



$$\int^{M} \hom_{C}(A, M) \times \hom(A, X) \times \hom_{C}(M \star Y, B)$$

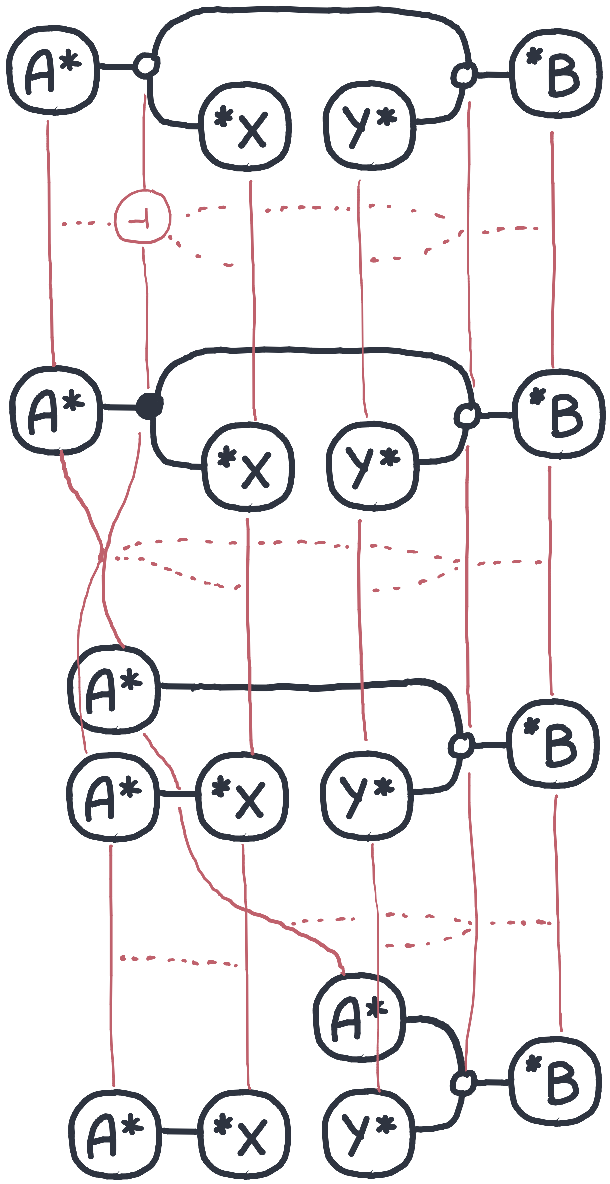
$$\cong$$



$$\hom(A, X) \times \hom_{C}(A \star Y, B)$$

Milewski, Profunctor optics: the categorical view.

Coend derivations



$$\int^{M} \hom_{C}(A, X \bullet M) \times \hom_{C}(M \bullet Y, B)$$

$\cong$ { Adjunction $\times \dashv \Delta$ }

$$\int^{M} \hom_{C\times C}(\Delta(A), X \bullet M) \times \hom_{C}(M \bullet Y, B)$$

$\cong$ { Definition of Δ }

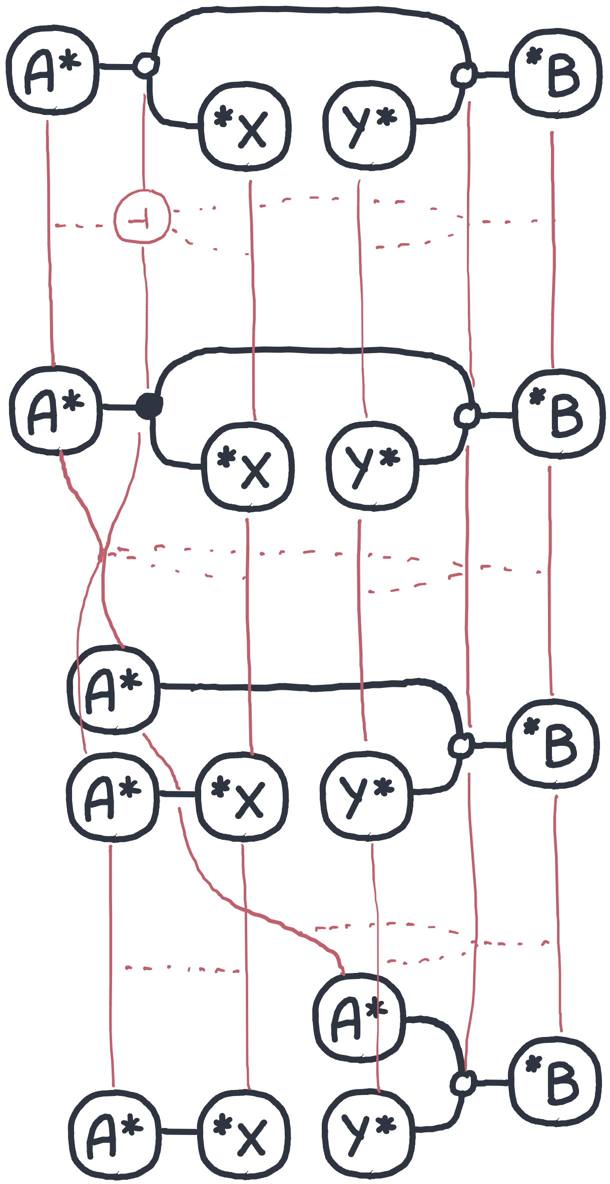
$$\int^{M} \hom_{C}(A, M) \times \hom(A, X) \times \hom_{C}(M \bullet Y, B)$$

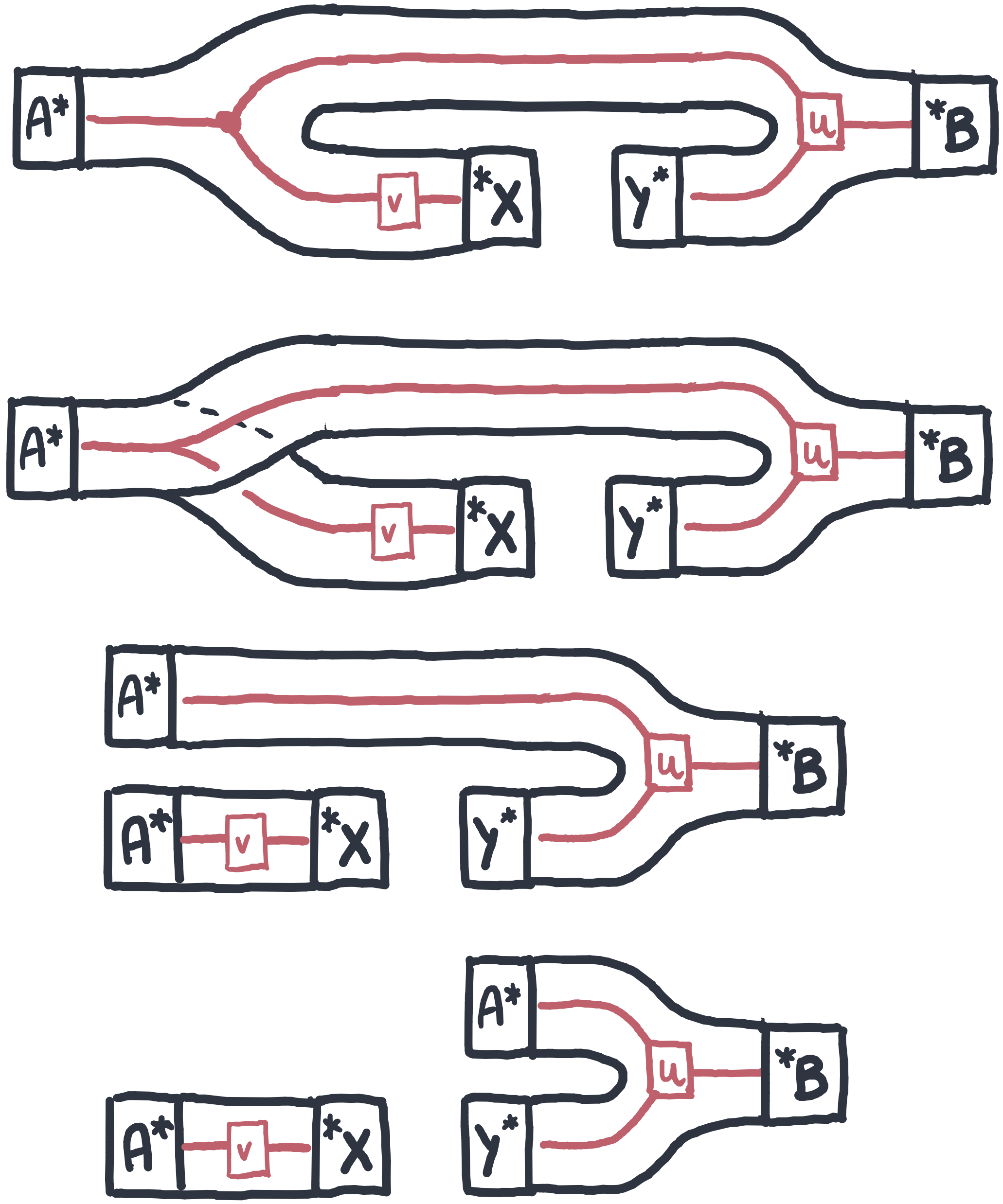
$\cong$ { Yoneda }

$$\hom(A, X) \times \hom_{C}(A \bullet Y, B)$$

Milewski, Profunctor optics: the categorical view.

Coend derivations

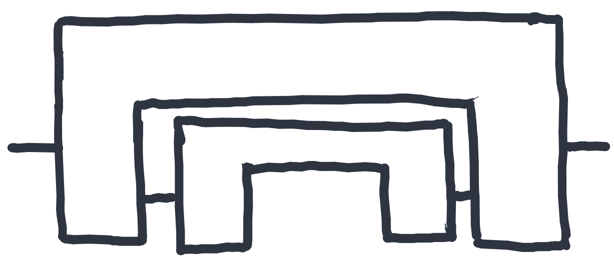




Lenses

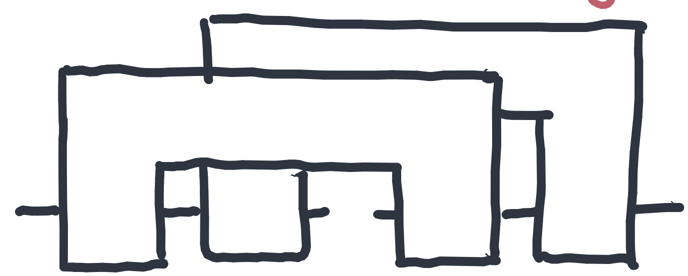
- Two ways of making lenses a category.

1



2

Requires symmetry.



Lenses

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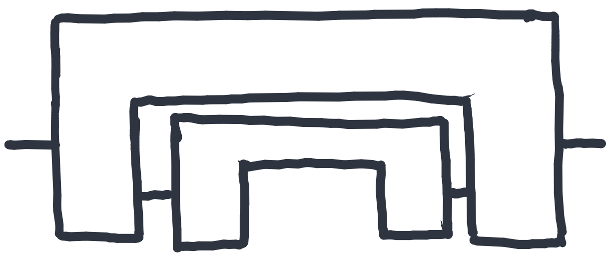

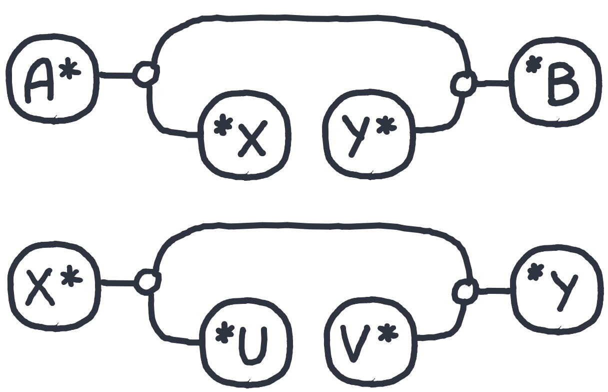

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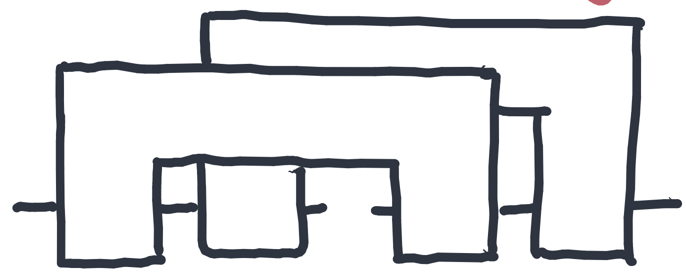

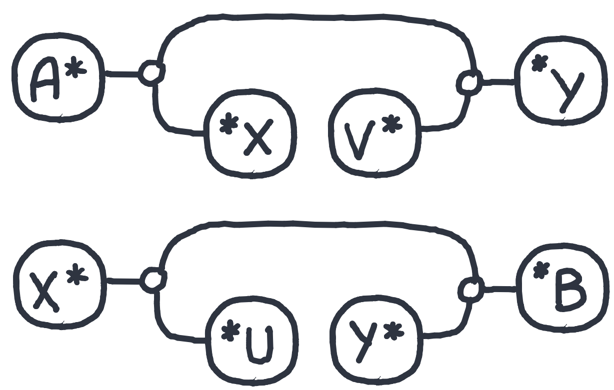

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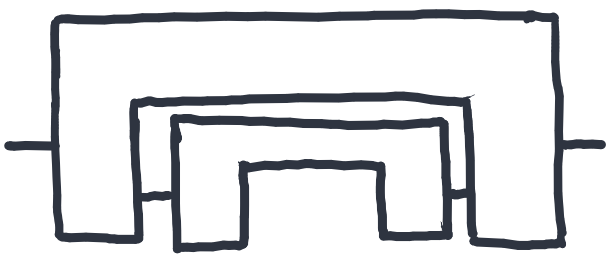
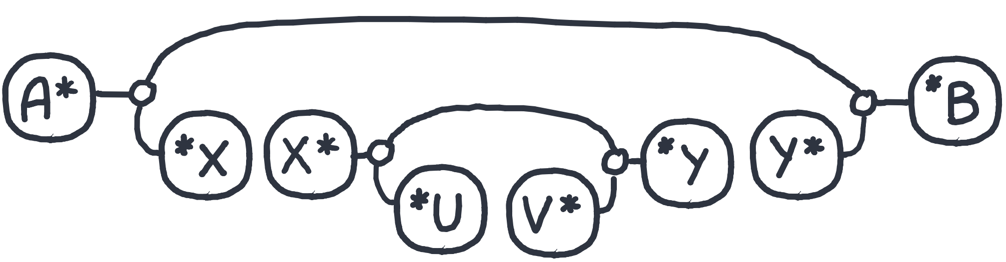
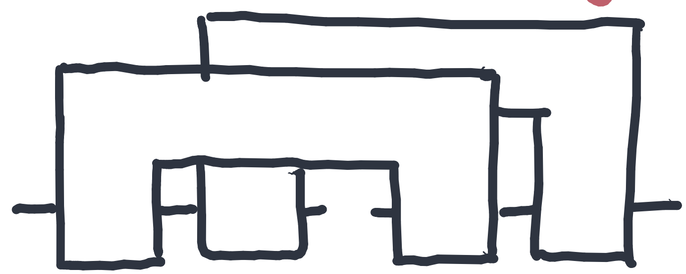
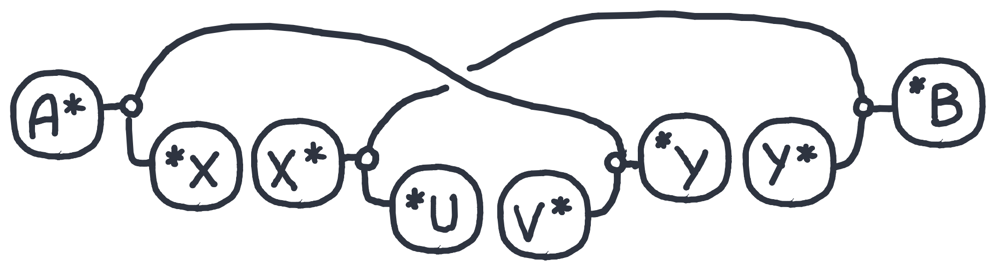




2

Requires Symmetry.





Lenses

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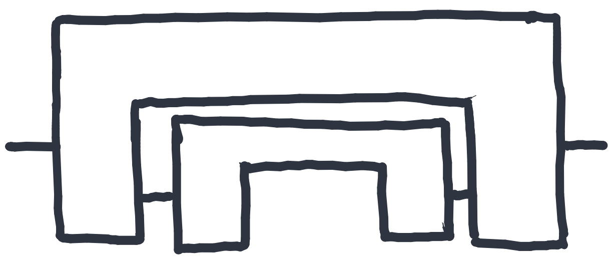

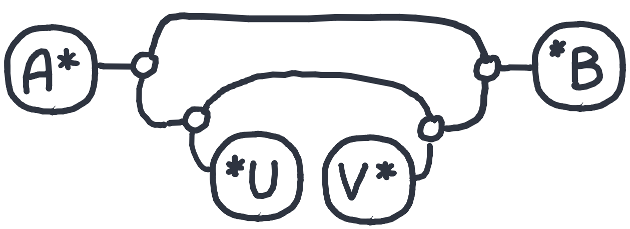

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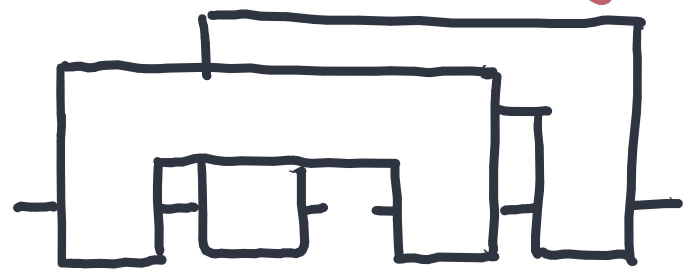

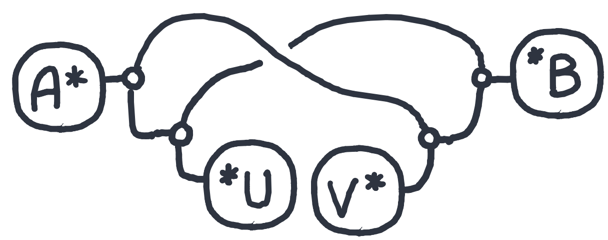

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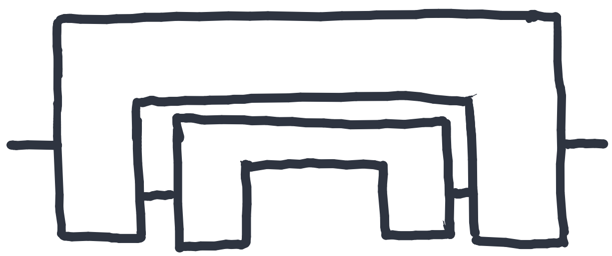

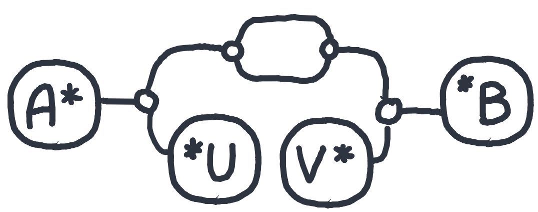

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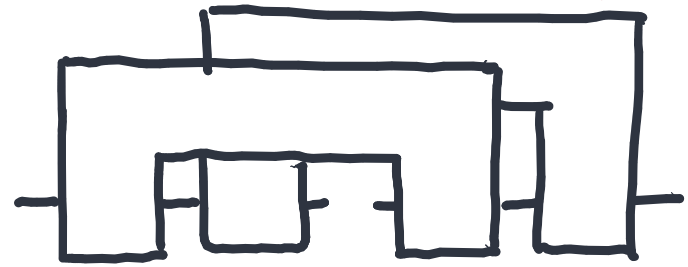
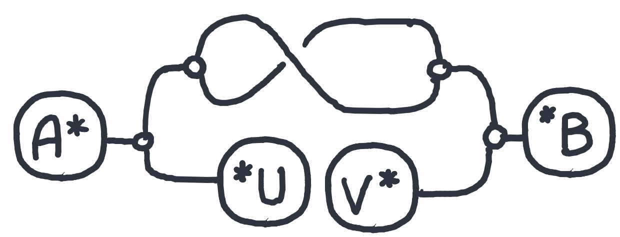
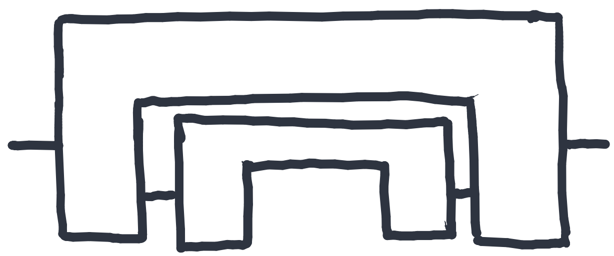


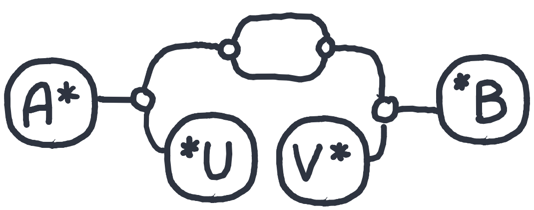


Lenses

- Two ways of making lenses a category.

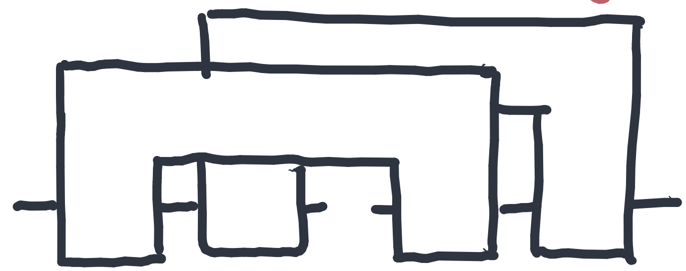
(1)

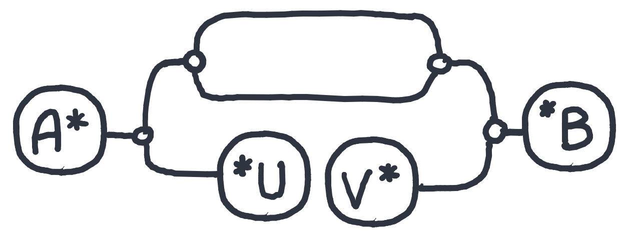




(2)

Requires Symmetry.





Lenses

- Two ways of making lenses a category.

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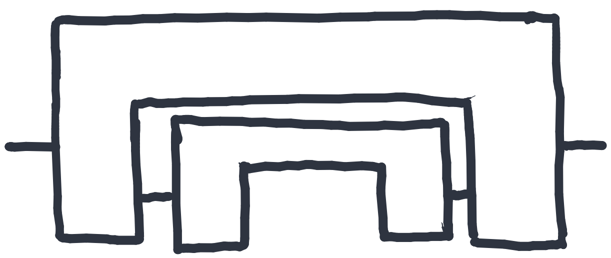
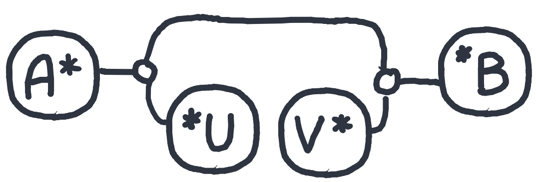
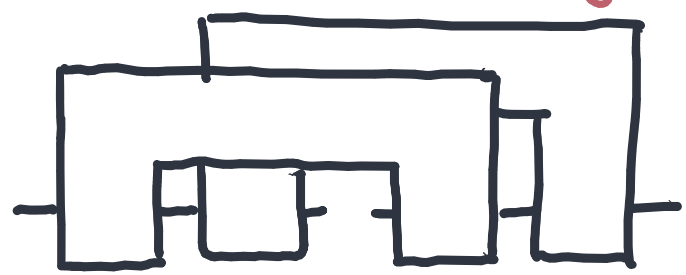
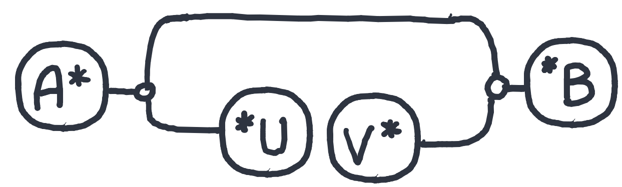




2

Requires Symmetry.





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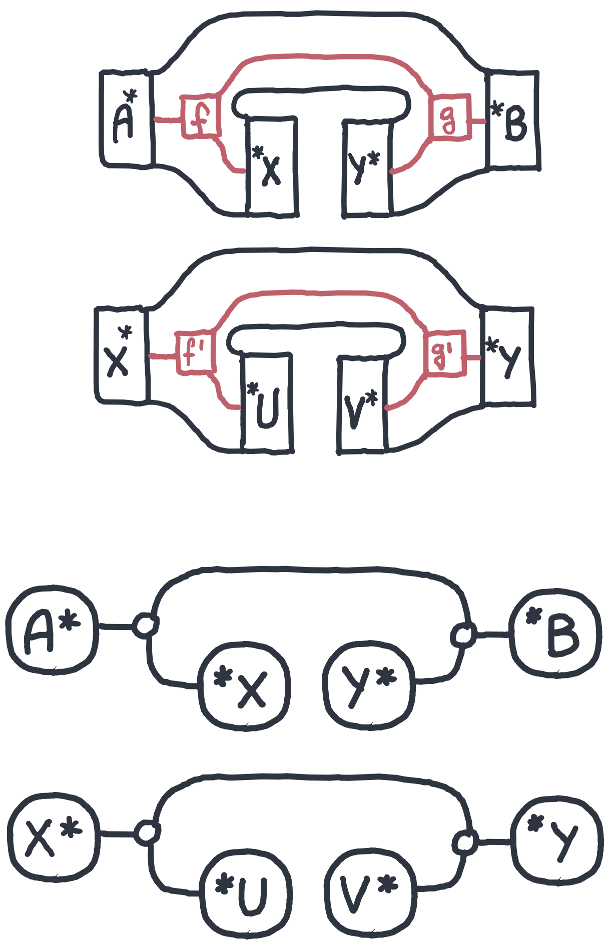

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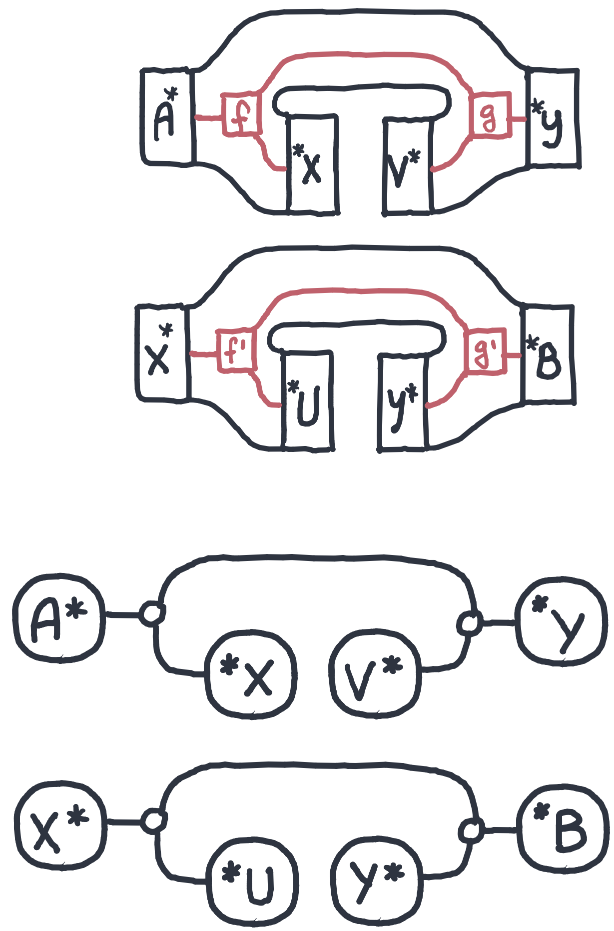

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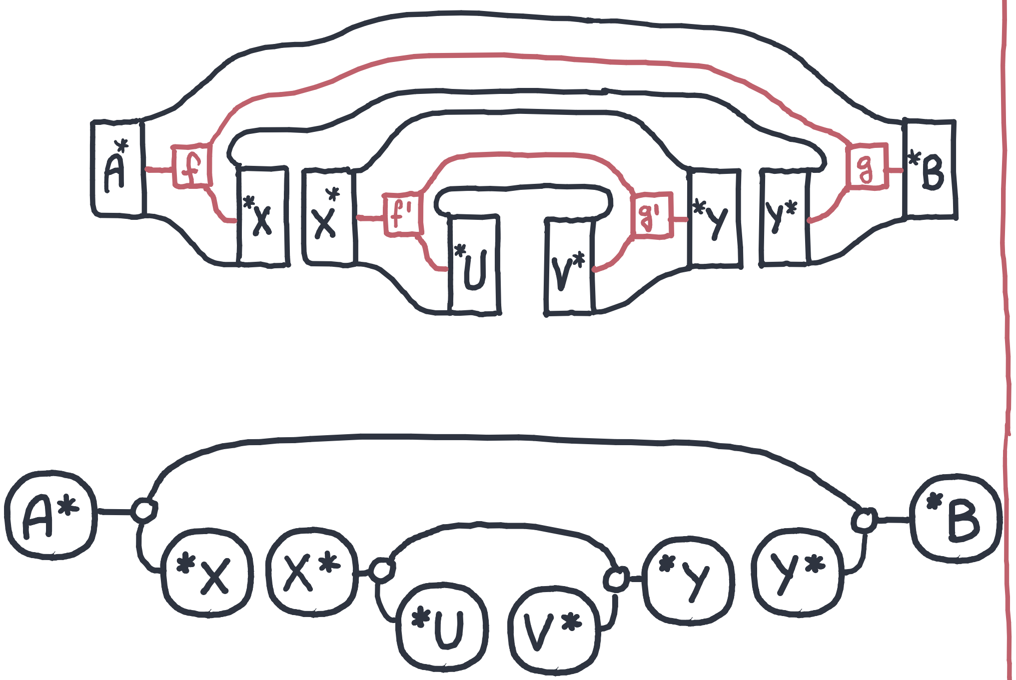

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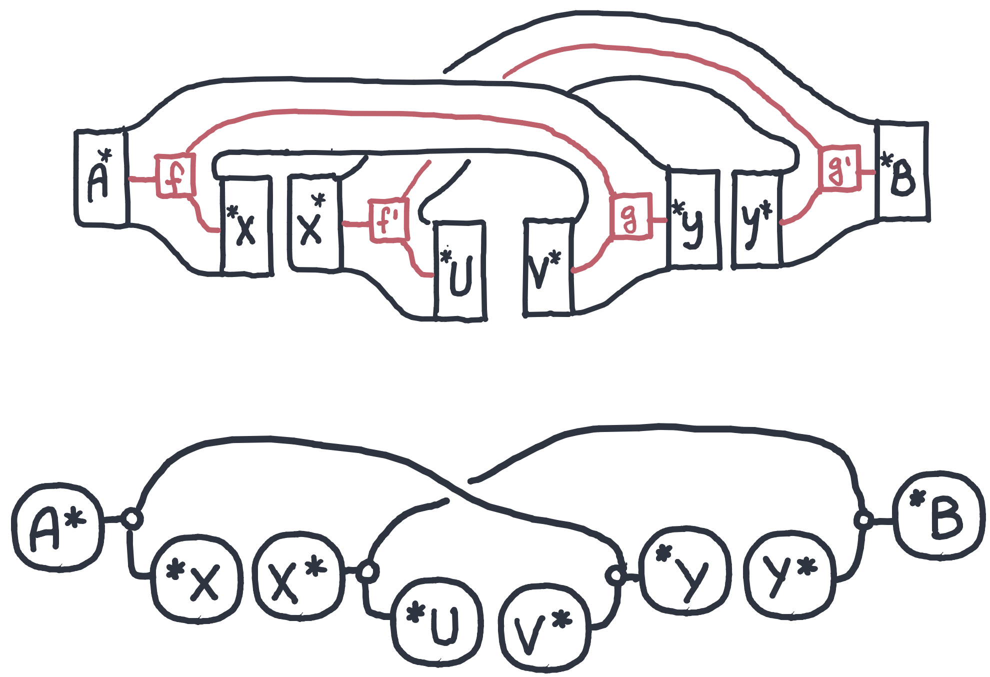

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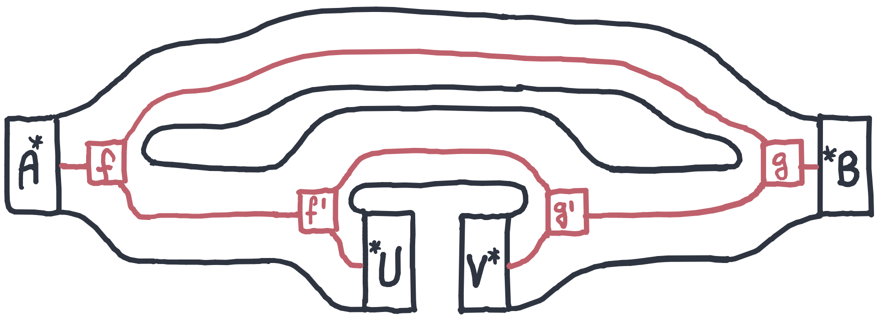

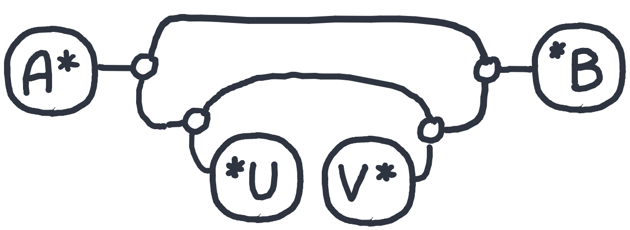

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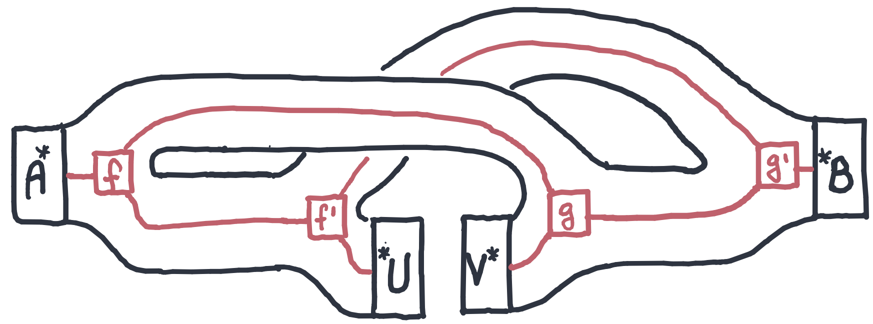

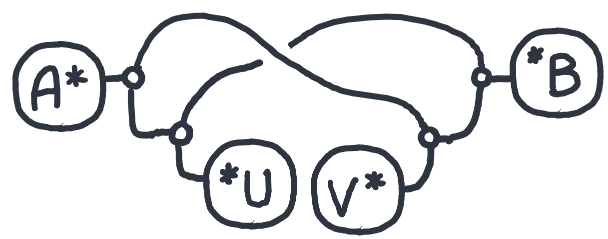

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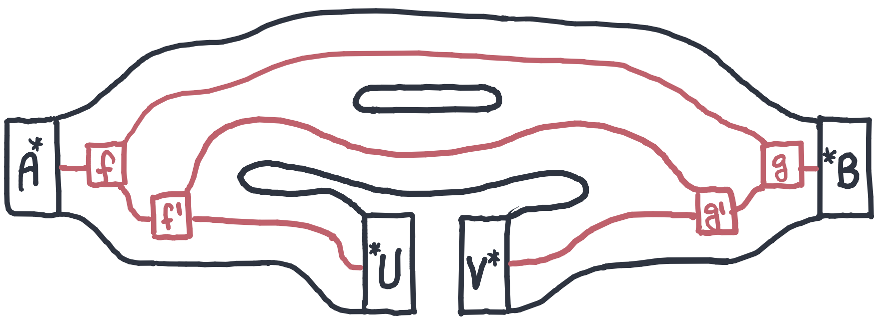

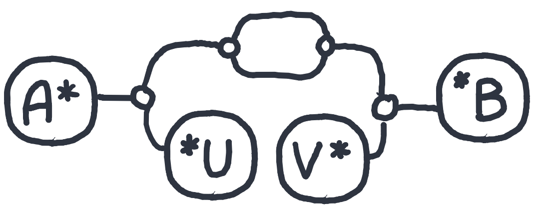

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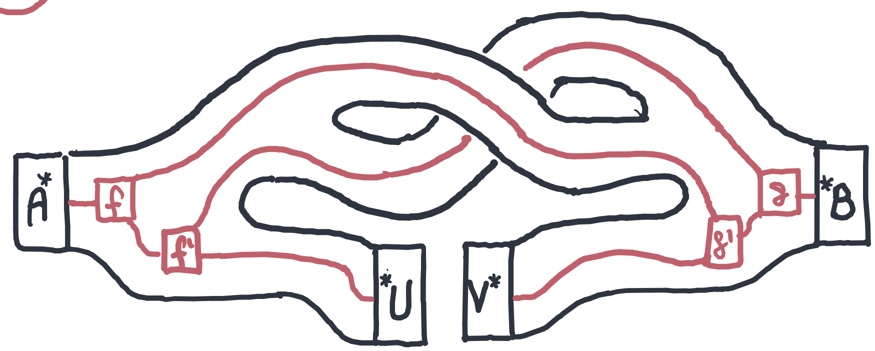

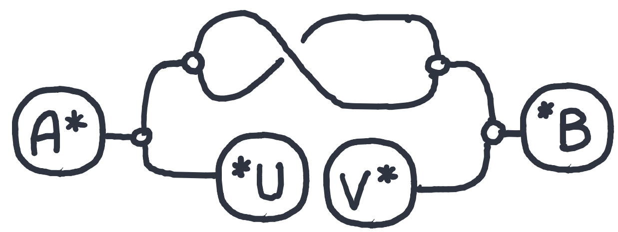

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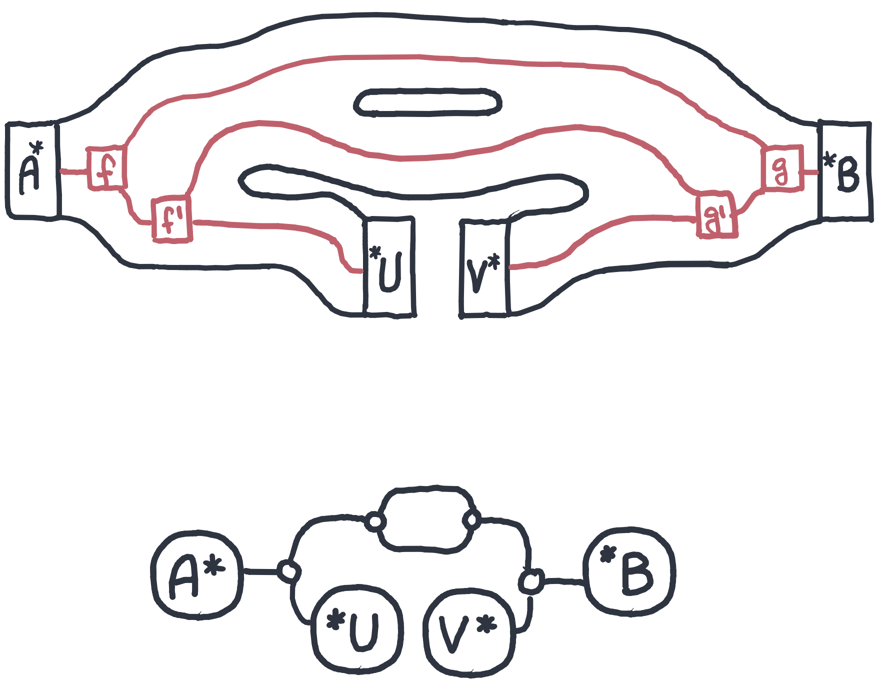

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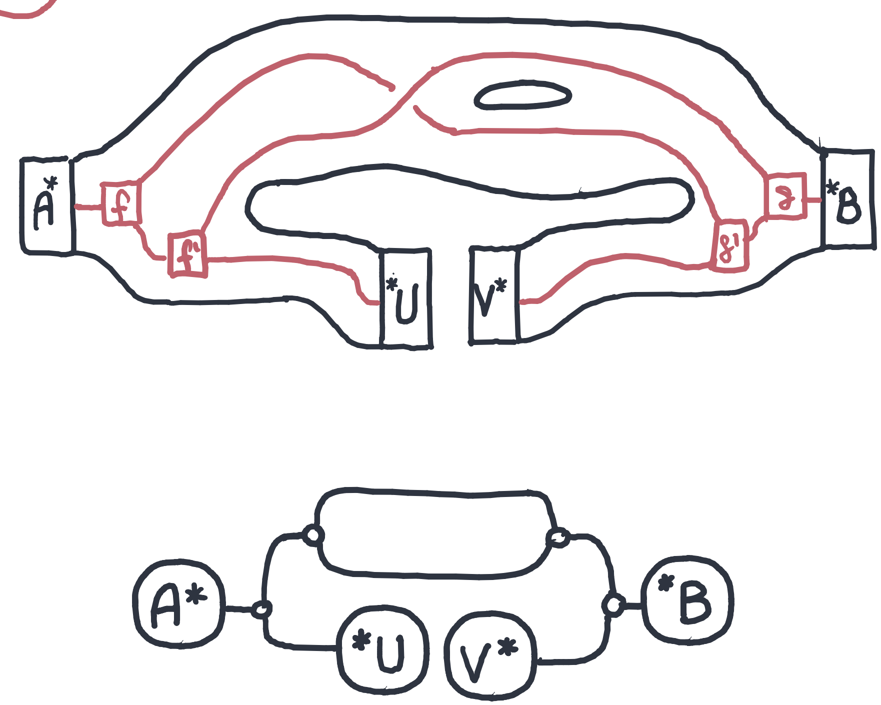

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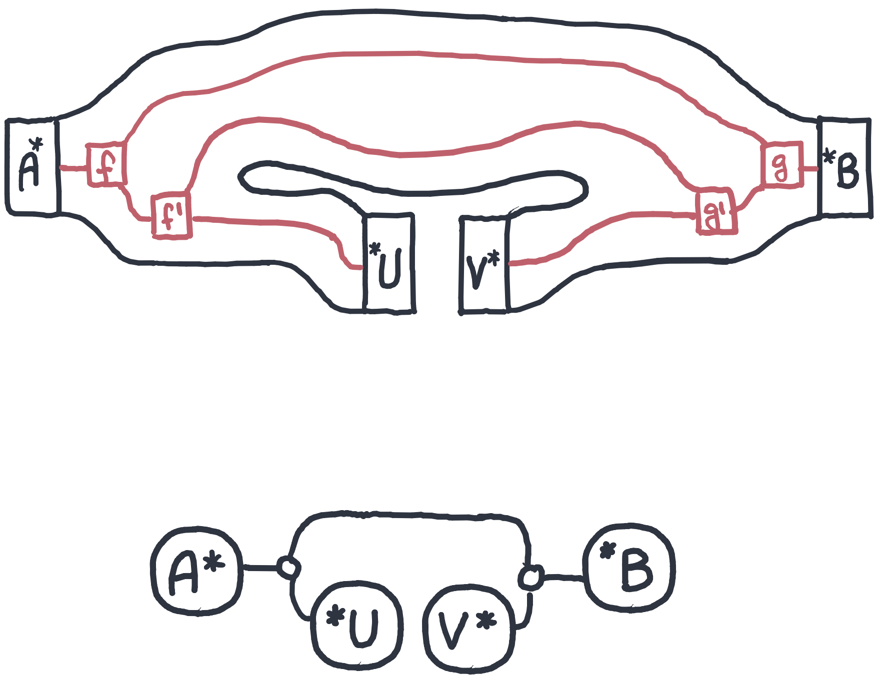

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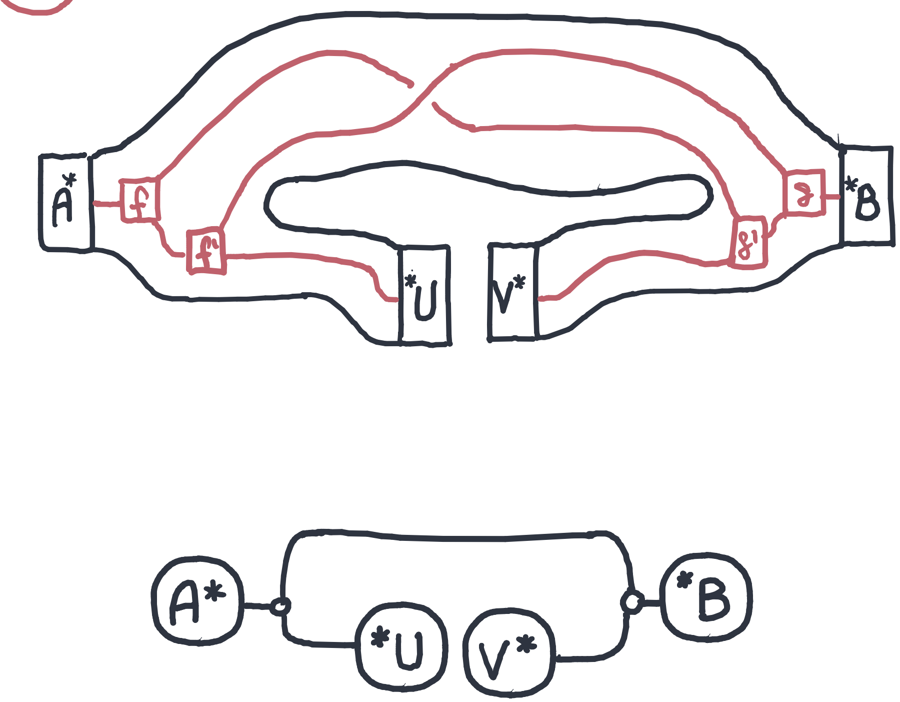

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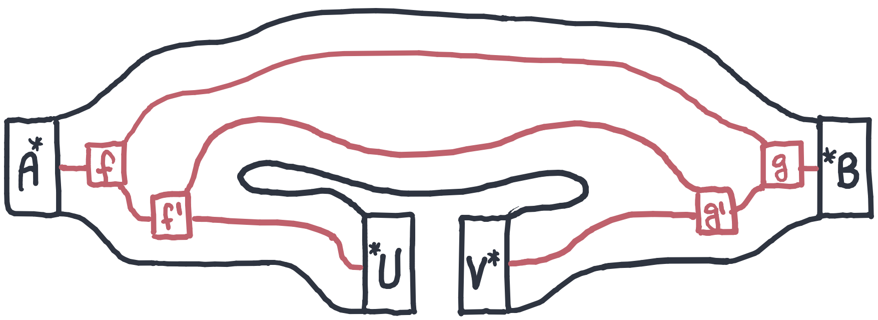

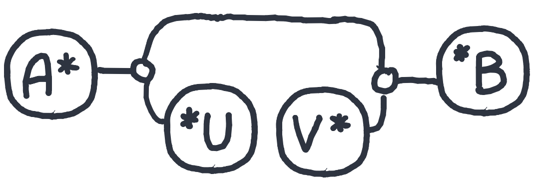

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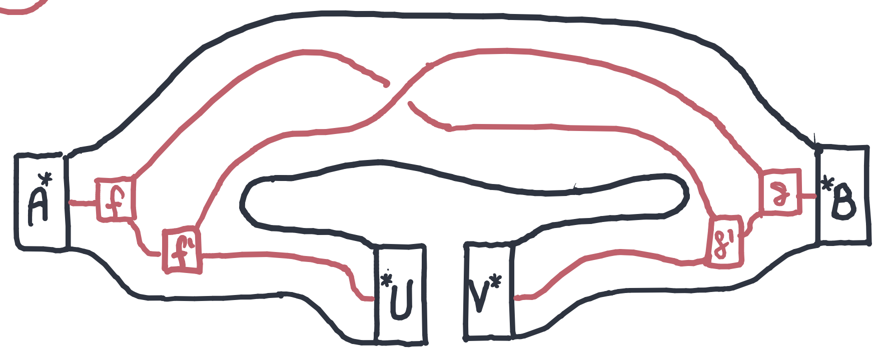
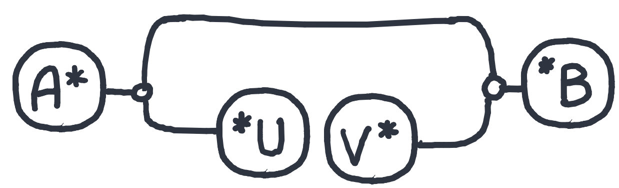
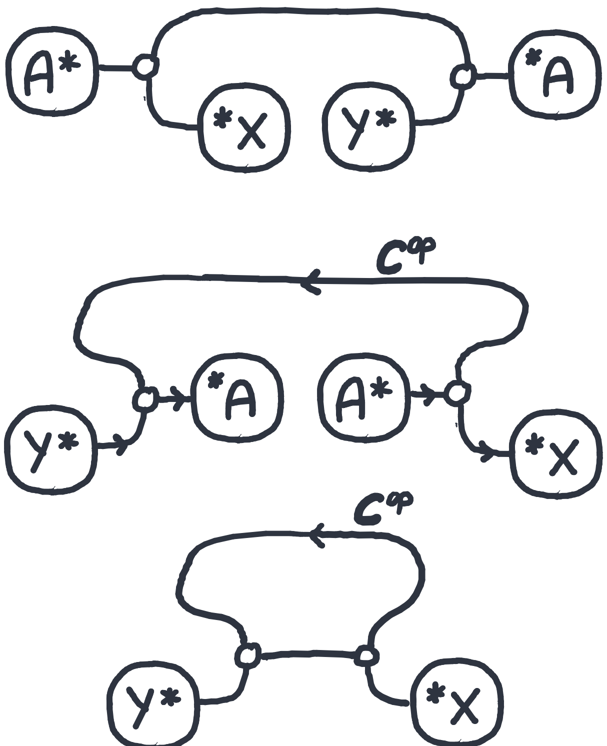


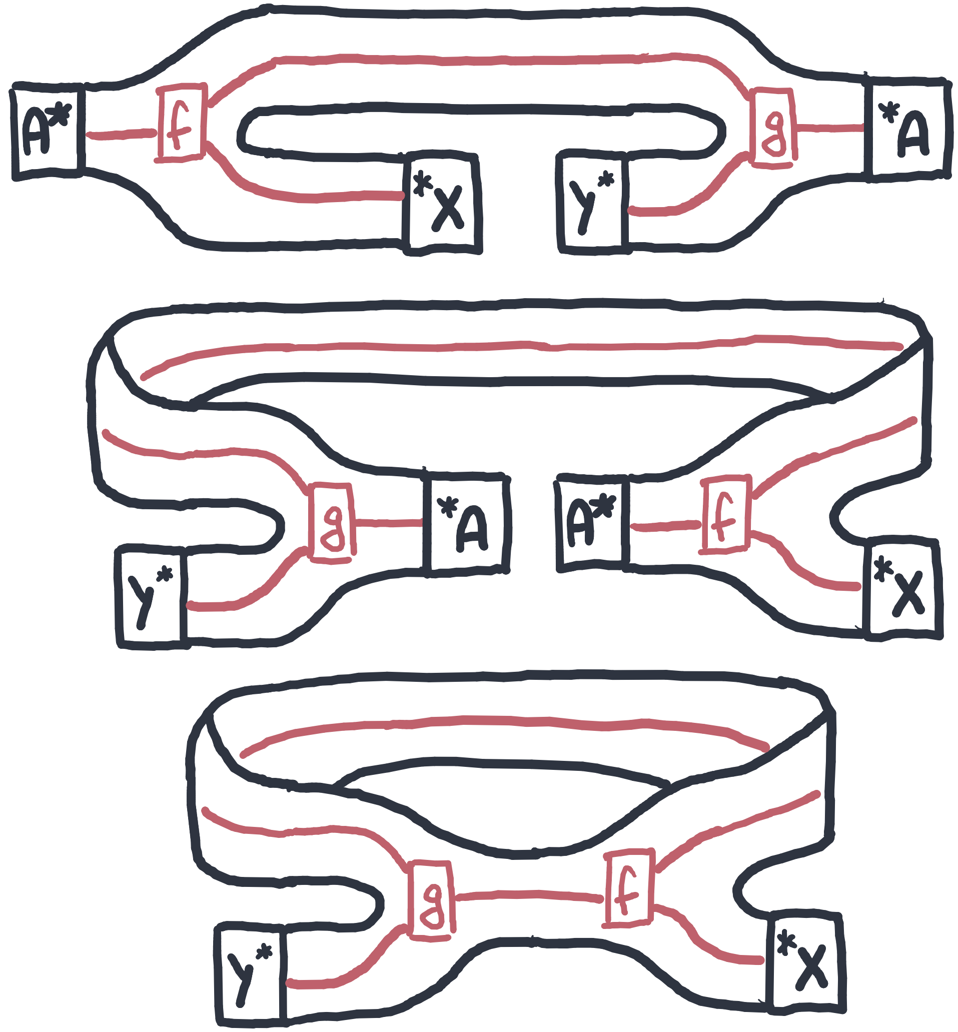


- Both are associative and unital.

Lenses and feedback

- A discrete dynamical system has the same data as a lens of certain type. [Schultz, Spivak, Vasilakopoulou]

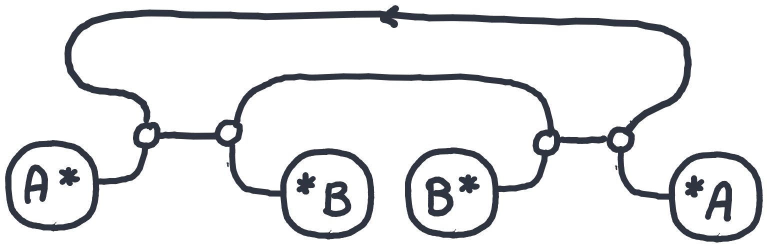




Morphism on the free category with feedback

Katis, Sabadini, Walters. Feedback, trace, and fixed-point semantics. (2002)

Learners



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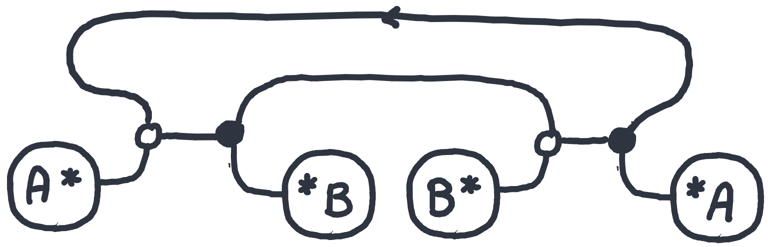

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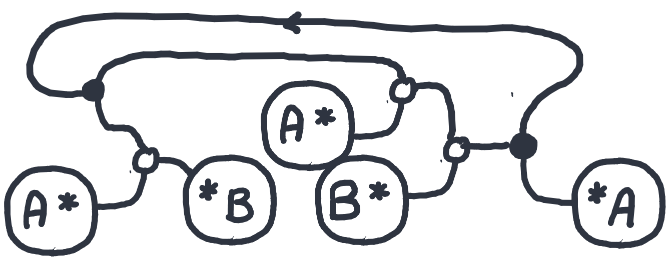

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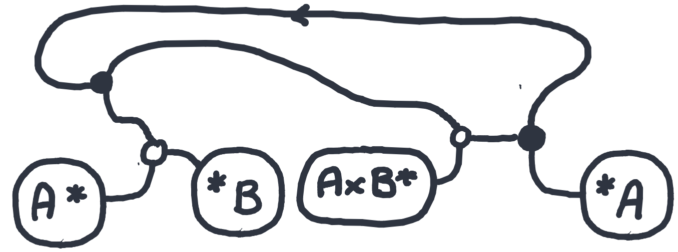

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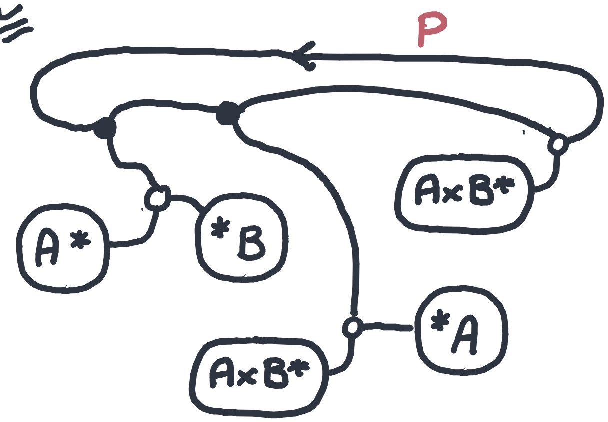


- A learner is given by

  $i : P \times A \to B$ (implementation)

  $r : P \times A \times B \to A$ (request)

  $u : P \times A \times B \to P$ (update)

- Fong, Spivak, Tuyéras. Backprop as a Functor: A compositional perspective on supervised learning. - Riley. Categories of optics.

References (Some, more on the paper)

Coend Calculus:

- Coend Calculus, Loregian.
- Categories for the working mathematician, MacLane.

Monoidal bicategories:

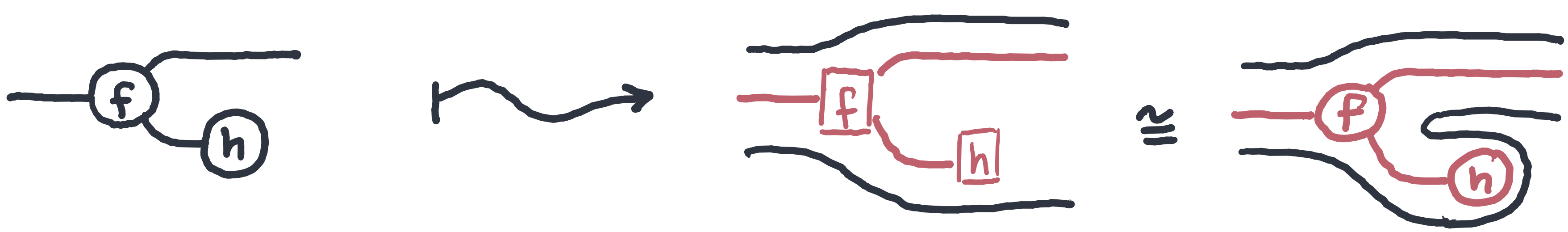
- Quasistrict Symmetric Monoidal 2-categories via Wire Diagrams, Bartlett.
- Modular categories as representations of the 3-dimensional bordism 2-category, Bartlett, Douglas, Schommer-Pries, Vicary.
- The classification of two-dimensional extended topological field theories, Schommer-Pries.
- Compact closed bicategories, Stay.
- Homotopy.io, Heidemann, Hu, Vicary.

the $\int_{A \in \mathbf{A}}$

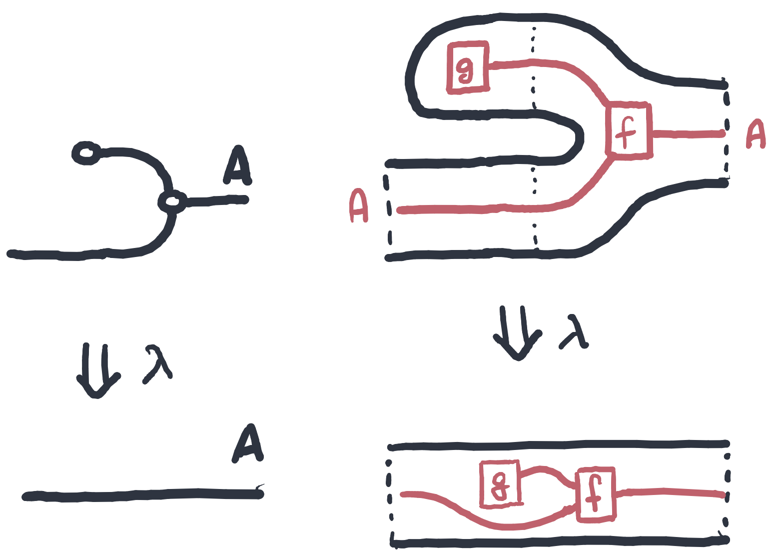
■ Email : mroman@ttu.ee , mromang08@gmail.com

Monoidal bicategory of pointed profunctors.

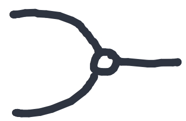
- String diagrams in a monoidal category can be lifted to $\mathbf{Prof}^*$:



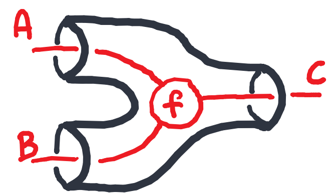
- Reductions in $\mathbf{Prof}$ can be lifted uniquely to $\mathbf{Prof}^*$:



Internal diagrams



$$\underset{\tilde{Z}^{A,B}_{C}}{\Longrightarrow}$$



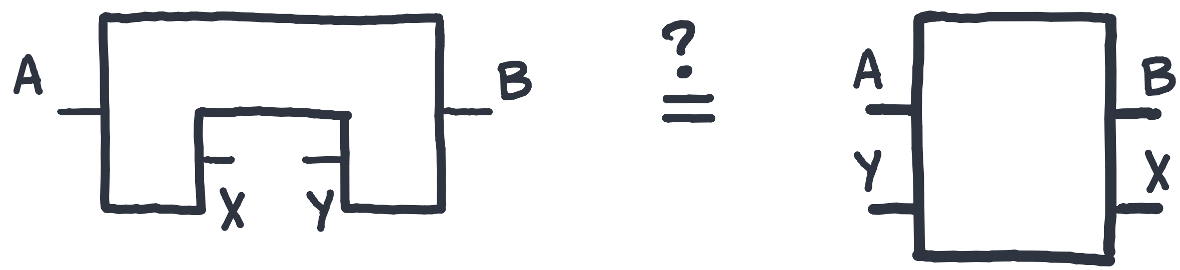
$\hom(- \otimes -, -)$ $\qquad\qquad$ $f \in \hom(A \otimes B, C)$

"We do not need to, and do not, make this notion of interior morphism geometrically precise (...) the pictures are merely a convenient mnemonic notation."

Bartlett, Douglas, Schommer-Pries, Vicary. Modular categories as representations of the 3-dimensional bordism 2-category. Hu. External traced monoidal categories.

Push to the boundaries

- Naive solution: push ports to the boundaries.



- Won't work, it does not preserve "time".

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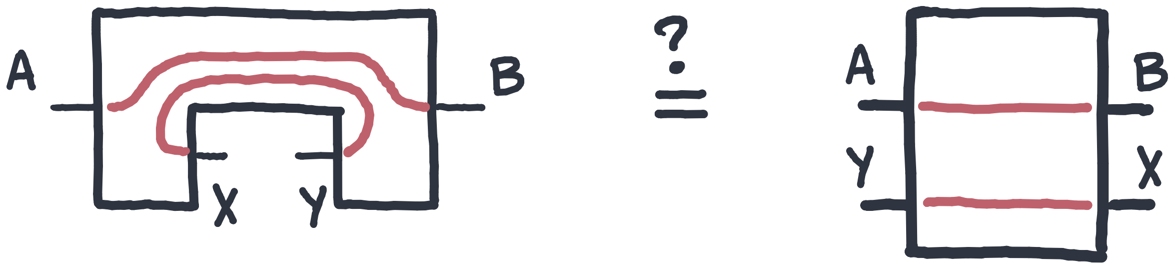


- Won't work, it does not preserve "time".

Ends (?)

- How to reason about ends?

$$\int_{A,B} \mathrm{Set}\big(\hom(A,B), P(A,B)\big) \cong P(A,A)$$

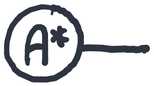
  - Many things work fine with this:

e.g. Tambara Module.

- If more is needed,
  - Closed structure of profunctors.
  - Completion and cocompletion (Isbell completion)

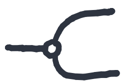
■ Willerton. Two 2-traces (Slides)

Notation

 : $1 \nrightarrow A$ $\quad$ $\hom(A, -)$ $\quad$ Represented objects.

 : $A \nrightarrow 1$ $\quad$ $\hom(-, A)$

 : $A \times A \nrightarrow A$ $\quad$ $\hom(1 \otimes 2, 3)$ $\quad$ Monoidal structure.

 : $A \nrightarrow A \times A$ $\quad$ $\hom(1, 2 \otimes 3)$

 : $A \times A \nrightarrow A$ $\quad$ $\hom(0, 1) \times \hom(0, 2)$ $\quad$ Copying-Discarding on Cat.

 : $A \nrightarrow A \times A$ $\quad$ $\hom(1, 0) \times \hom(2, 0)$