

A Canonical Algebra of Open Transition Systems

Elena Di Lavore, Alessandro Gianola, Mario Román, Nicoletta Sabadini
and Paweł Sobociński

Applied Category Theory 2021
University of Cambridge.

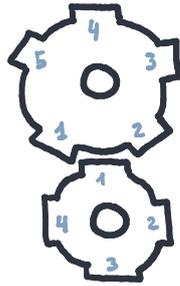
arXiv: 2010.10069

PART 1:

SPANS OF GRAPHS

Span (Graph), algebra of open transition systems

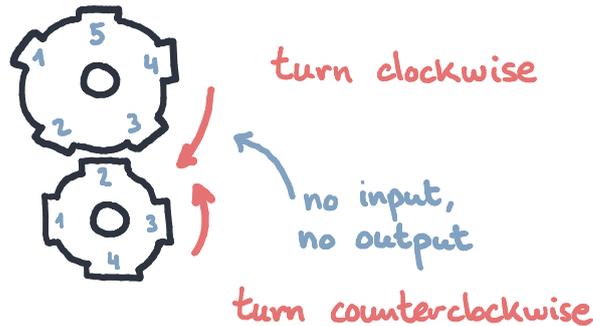
Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.



no input,
no output

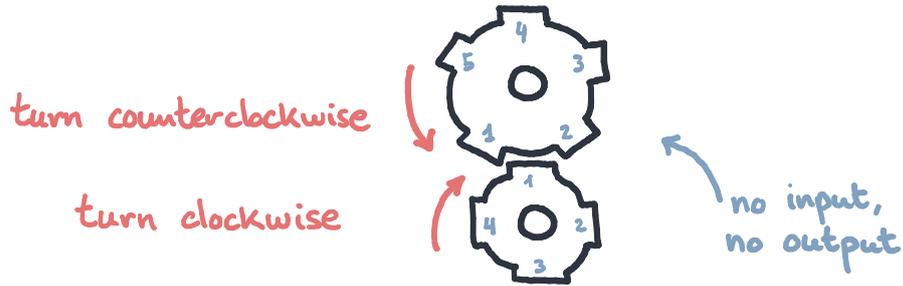
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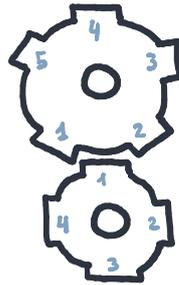
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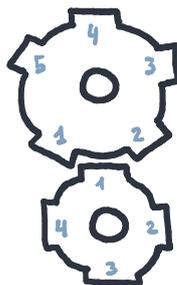


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How to model this situation?

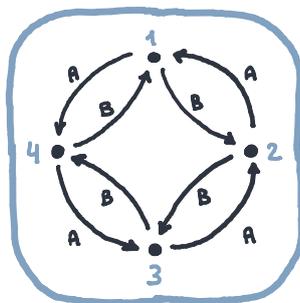
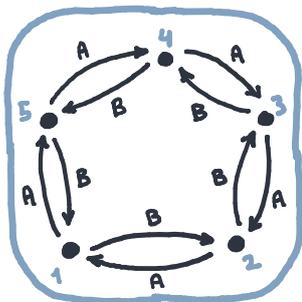
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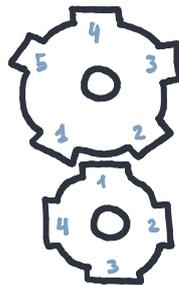
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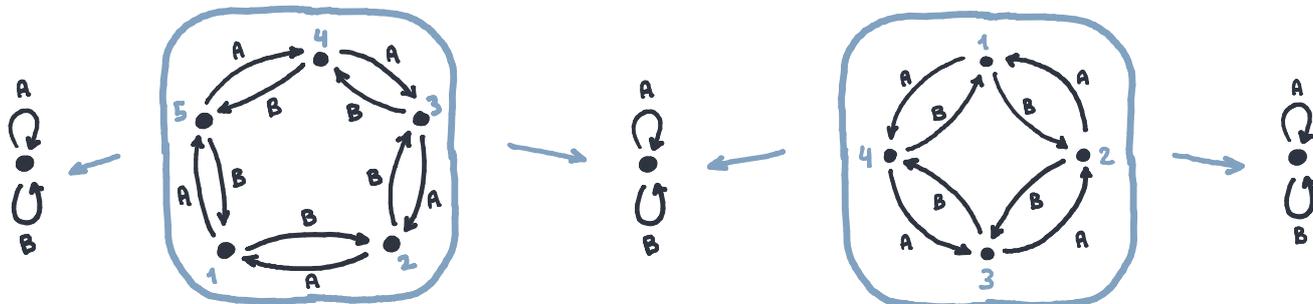
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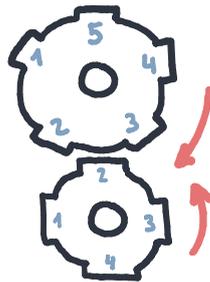
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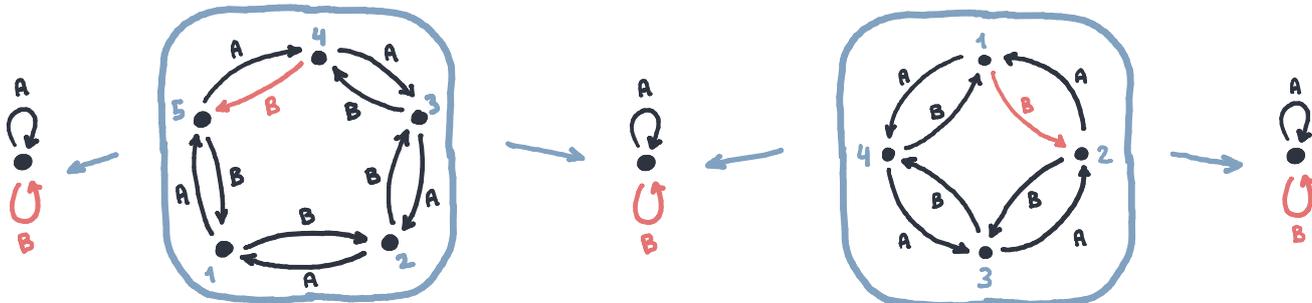
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turn clockwise

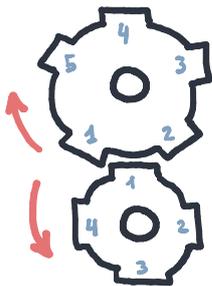
turn counterclockwise

How to model this situation?

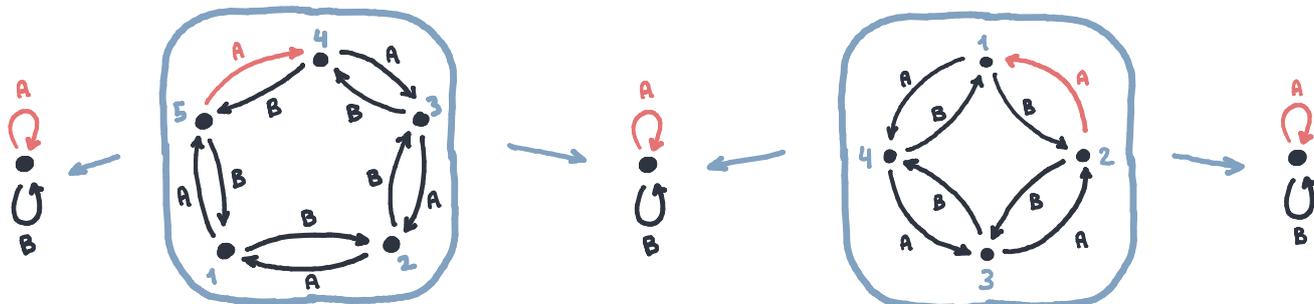


Span (Graph), algebra of open transition systems

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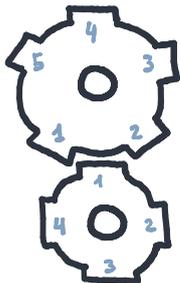


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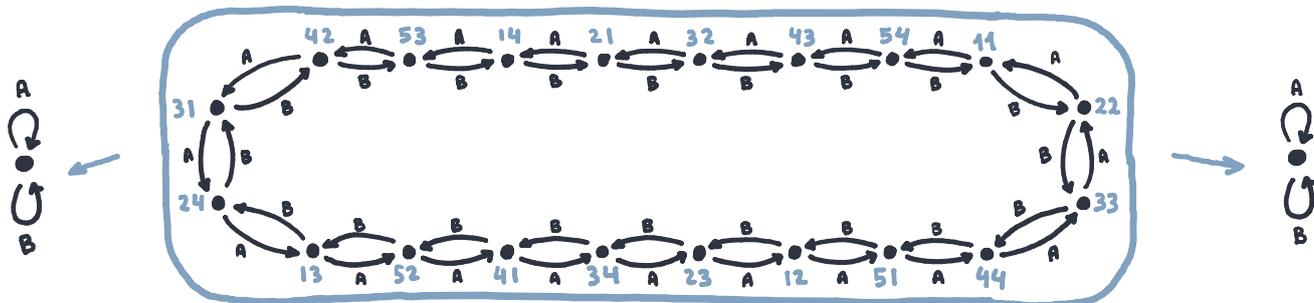


Span (Graph), algebra of open transition systems

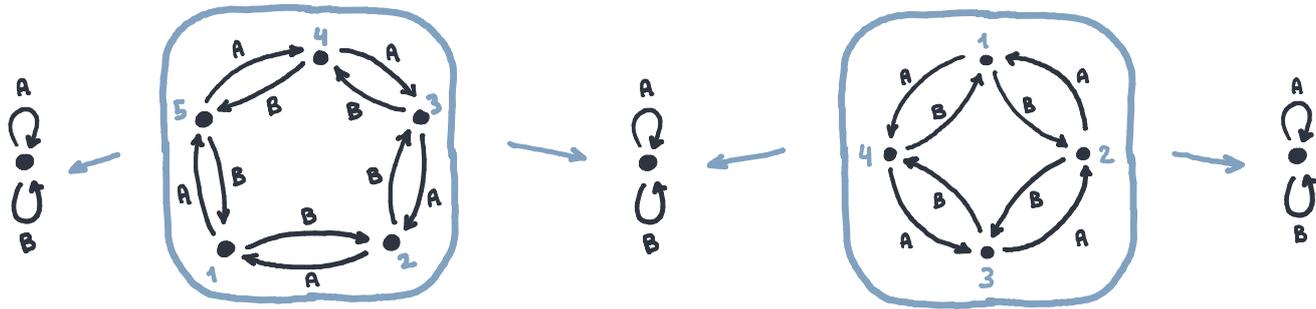
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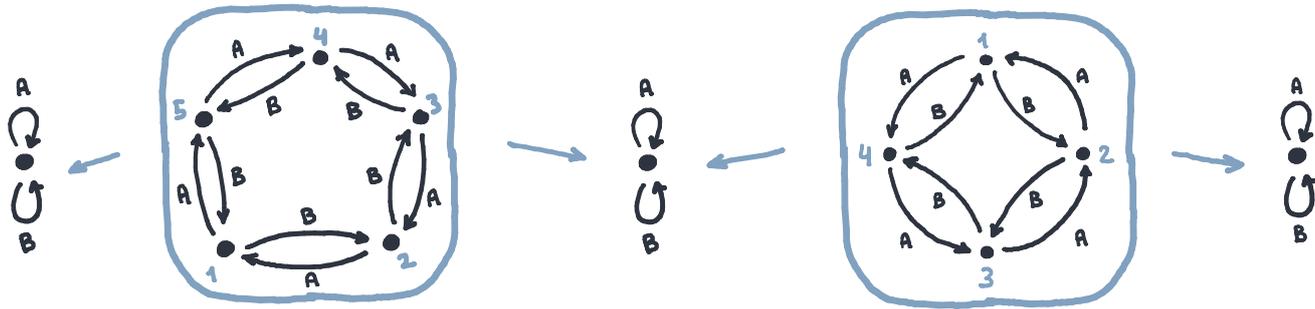
Span(Graph), algebra of open transition systems



SPAN(GRAPH):

- Compositional, stateful transition systems.
- Synchronization by composition.
- Transition systems are encoded as graphs.
- Boundaries may be single-vertex graphs, $\text{SPAN}(\text{GRAPH})_*$.

Span(Graph), algebra of open transition systems



SPAN(GRAPH):

- Compositional, stateful transition systems.
 - Synchronization by composition.
 - Transition systems are encoded as graphs.
 - Boundaries may be single-vertex graphs, $\text{SPAN}(\text{GRAPH})_*$.
- Ad hoc?*

PART 2:

STATEFUL MORPHISMS

The $\text{St}(\cdot)$ construction

DEFINITION. For $(\mathbb{C}, \otimes, \mathbb{I})$ symmetric monoidal,

$$\text{St}(\mathbb{C})(A, B) := \left\{ (S, \varphi) \mid S \in \text{ob } \mathbb{C}, \varphi: S \otimes A \longrightarrow S \otimes B \right\} / \sim,$$

↖ stateful morphism
↘ state space

quotiented by the equivalence relation

$$\left(S \mid \begin{array}{c} S \\ A \end{array} \begin{array}{|c} \varphi \\ \hline \end{array} \begin{array}{c} S \\ B \end{array} \right) \sim \left(T \mid \begin{array}{c} T \\ A \end{array} \begin{array}{|c} \gamma' \\ \hline \end{array} \begin{array}{|c} \varphi \\ \hline \end{array} \begin{array}{|c} \gamma \\ \hline \end{array} \begin{array}{c} S \\ B \\ T \end{array} \right)$$

where $\gamma: S \cong T$ is any isomorphism.

Diagrammatic algebra: from linear to concurrent systems. Bonchi, Holland, et al.
Memoryful geometry of interaction. Hoshino, Muroya, Hasuo.
Differentiable Causal Computations via Delayed Trace. Katsumata, Sprunger.

The $St(\cdot)$ construction

Composition is given by:

$$\left(S \left| \begin{array}{c} S \\ A \end{array} \right| \Psi \left| \begin{array}{c} S \\ B \end{array} \right. \right) \circ \left(T \left| \begin{array}{c} T \\ B \end{array} \right| \Psi \left| \begin{array}{c} T \\ C \end{array} \right. \right) = \left(S \otimes T \left| \begin{array}{c} S \\ T \\ A \end{array} \right| \Psi \left| \begin{array}{c} S \\ T \\ B \end{array} \right. \right) \Psi \left| \begin{array}{c} S \\ T \\ C \end{array} \right. \right).$$

Tensoring is given by:

$$\left(S \left| \begin{array}{c} S \\ A \end{array} \right| \Psi \left| \begin{array}{c} S \\ B \end{array} \right. \right) \otimes \left(S' \left| \begin{array}{c} S' \\ A' \end{array} \right| \Psi' \left| \begin{array}{c} S' \\ B' \end{array} \right. \right) = \left(S \otimes S' \left| \begin{array}{c} S' \\ S \\ A' \\ A \end{array} \right| \Psi \left| \begin{array}{c} S' \\ S \\ B' \\ B \end{array} \right. \right) \Psi' \left| \begin{array}{c} S' \\ S \\ B' \\ B \end{array} \right. \right).$$

Universal property?

Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.

The $St(\cdot)$ construction

$ST(SET_x)$: $S \times A \rightarrow S \times B$ Mealy transition system

$ST(SET_+)$: $S + A \rightarrow S + B$ Elgot transition system

$ST(REL_x)$: $S \times A \rightarrow \mathcal{P}(S \times B)$ Non-deterministic transition system

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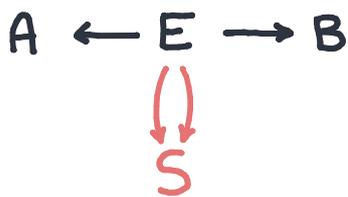
$ST(SPAN(SET))$: $S \times A \leftarrow E \rightarrow S \times B$

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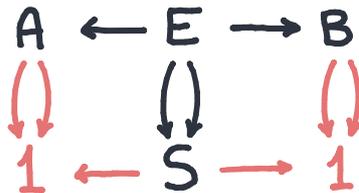
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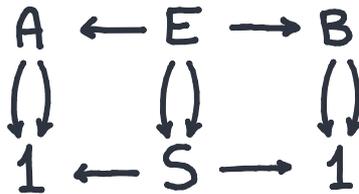
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$SPAN(GRAPH)_*$

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$ST(SPAN(SET))$:
$$\begin{array}{ccccc} A & \leftarrow & E & \rightarrow & B \\ \Downarrow & & \Downarrow & & \Downarrow \\ 1 & \leftarrow & S & \rightarrow & 1 \end{array} \quad SPAN(GRAPH)_*$$

THEOREM. There is a monoidal isomorphism:

$$ST(SPAN(SET)) \cong SPAN(GRAPH)_*$$

stateful synchronization: spans of graphs

PART 3:

FEEDBACK

Categories with feedback

Symmetric monoidal category with an operator

$$\text{fbk}_S : \text{hom}(S \otimes A, S \otimes B) \longrightarrow \text{hom}(A, B),$$

such that:

$$\textcircled{1} \quad u; \text{fbk}_S(f); v = \text{fbk}_S((u \otimes \text{id}); f; (v \otimes \text{id}))$$

$$\textcircled{2} \quad \text{fbk}_I(f) = f$$

$$\textcircled{3} \quad \text{fbk}_S(\text{fbk}_T(f)) = \text{fbk}_{S \otimes T}(f)$$

$$\textcircled{4} \quad \text{fbk}_S(f) \otimes g = \text{fbk}_S(f \otimes g)$$

$$\textcircled{5} \quad \text{fbk}(f; (h \otimes \text{id})) = \text{fbk}((h \otimes \text{id}); f)$$

Feedback, trace, and fixed-point semantics.

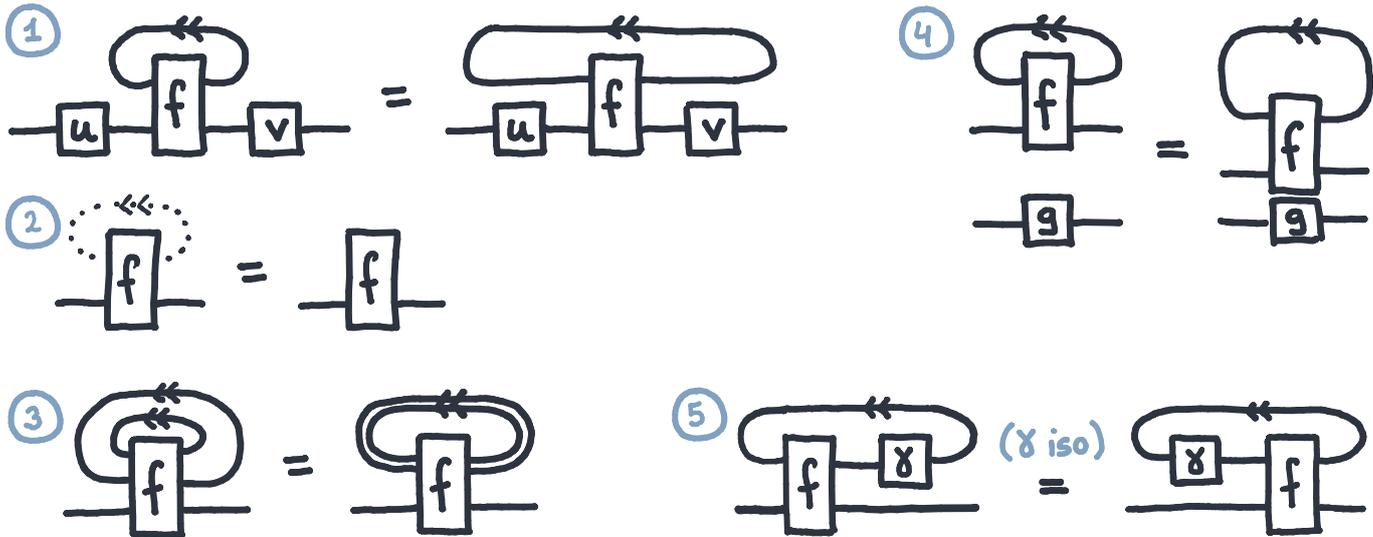
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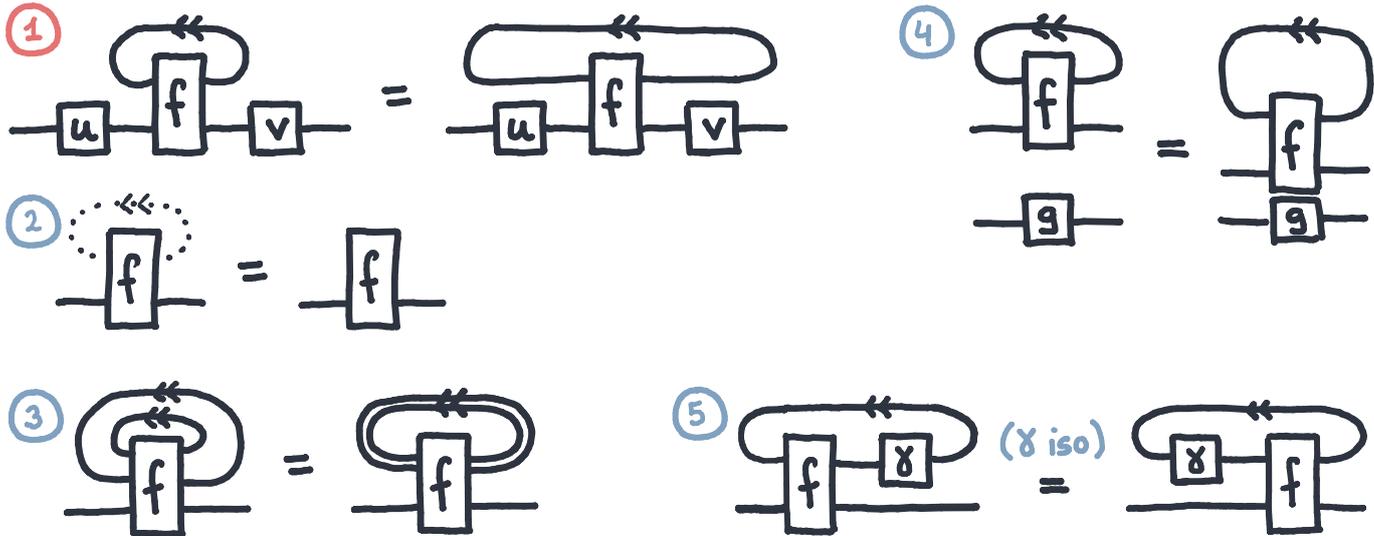
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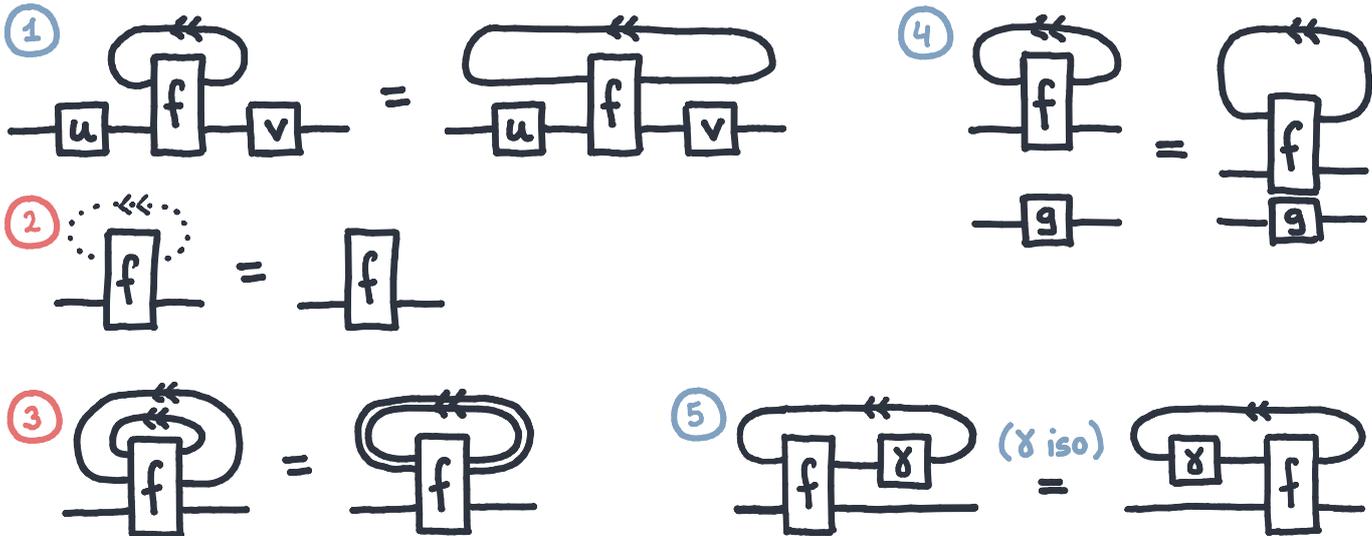
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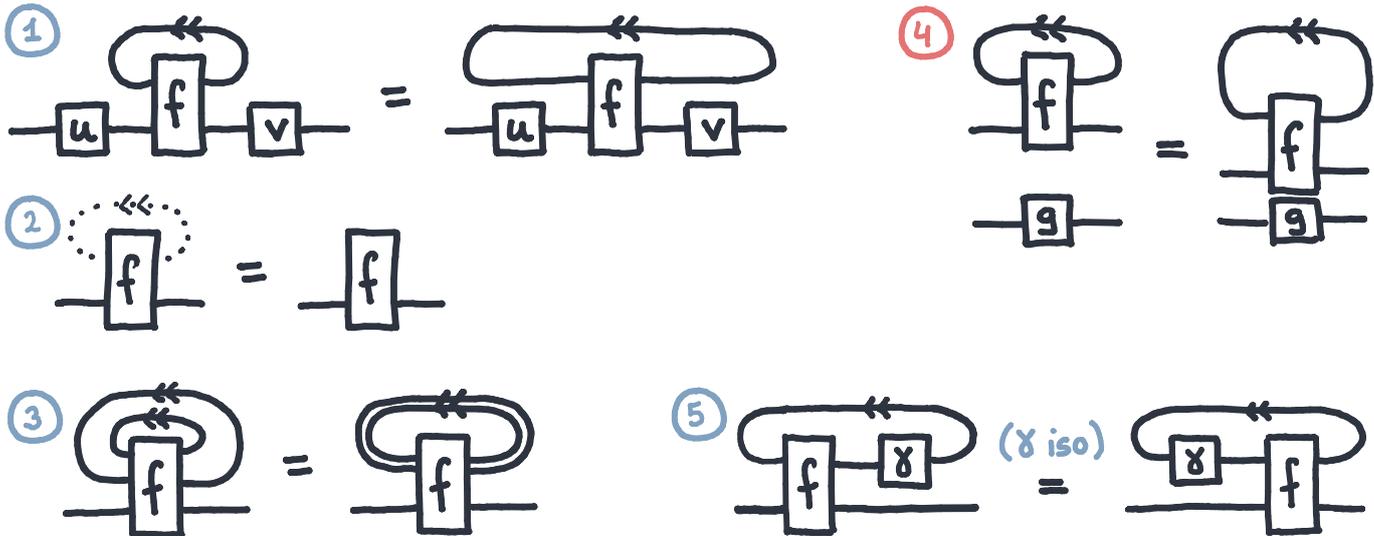
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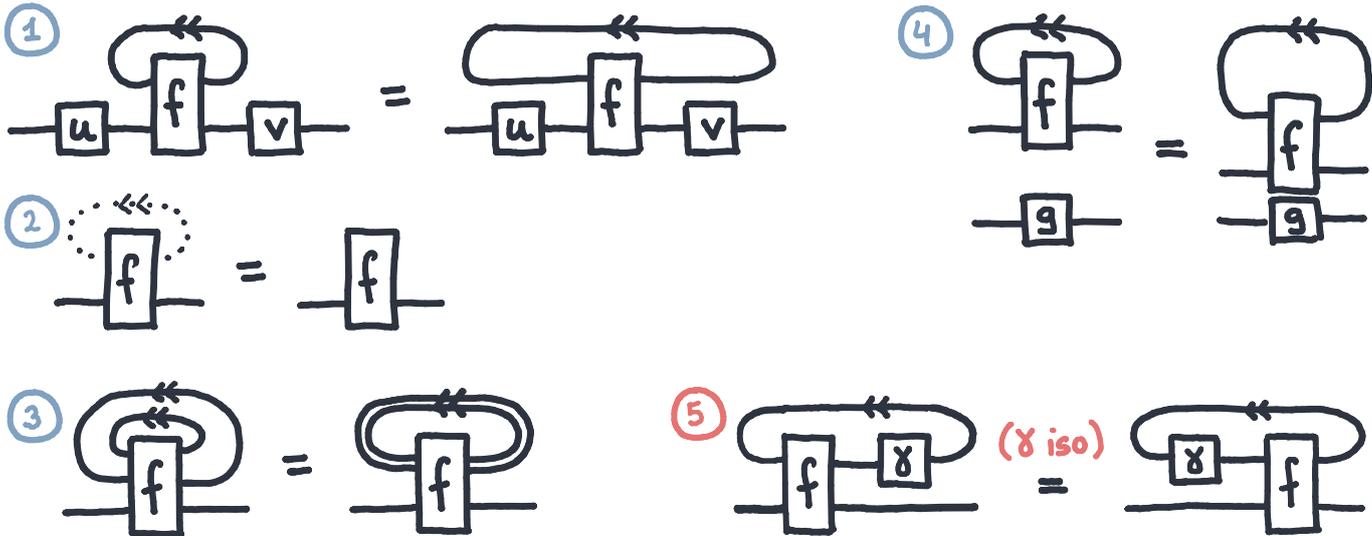
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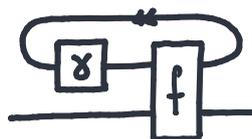
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Categories with feedback

Differences with traced monoidal categories?



≠



(WEAKSLIDING)



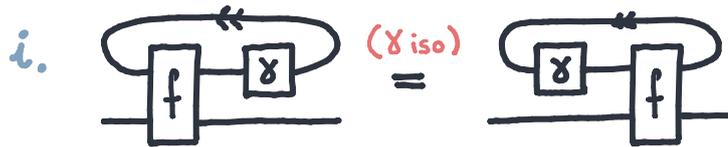
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(YANKING)

Categories with feedback

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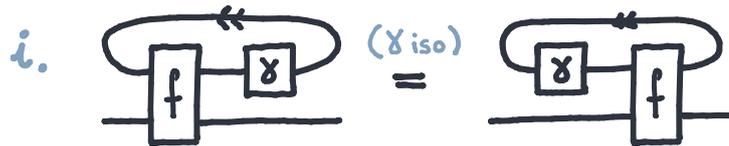
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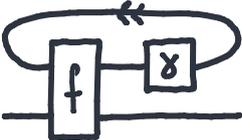
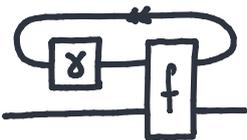
(WEAKSLIDING)



(YANKING)

Categories with feedback

Differences with traced monoidal categories?

i.  $(\gamma \text{ iso}) =$ 

(WEAKSLIDING)

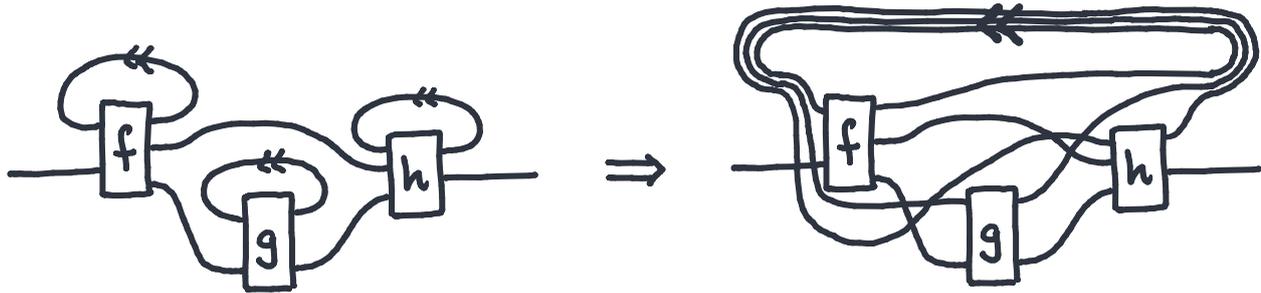
ii.  $:=$ 

(YANKING)

- Feedback is weaker than trace (and balanced trace).
- Feedback and guarded trace coincide in compact closed categories.
- Feedback has a different type than delayed trace.

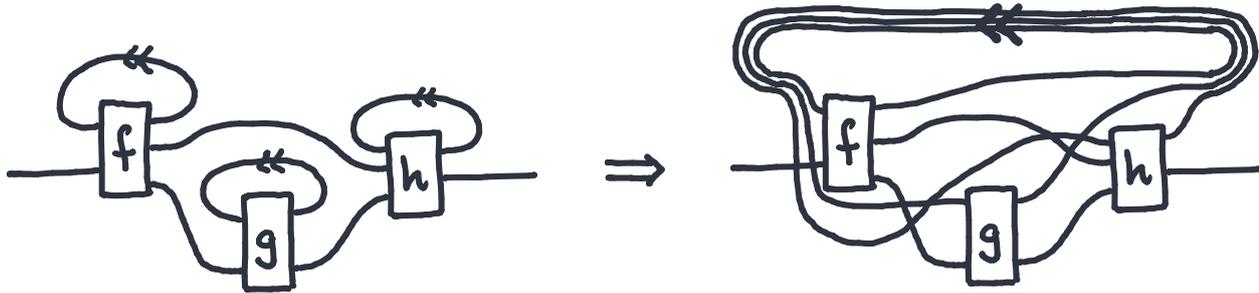
Categories with feedback

Multiple applications of feedback can be reduced into a single one. All of the axioms are needed for this result.

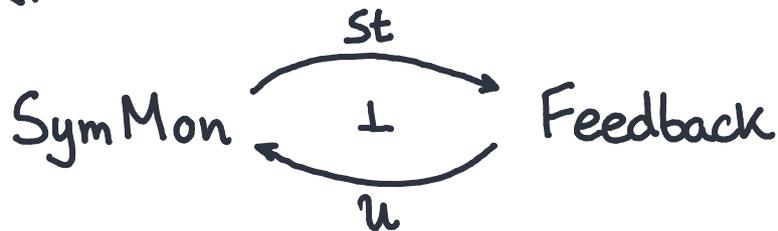


Categories with feedback

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We can use this to show that $St(\mathbb{C})$ is the free category with feedback.

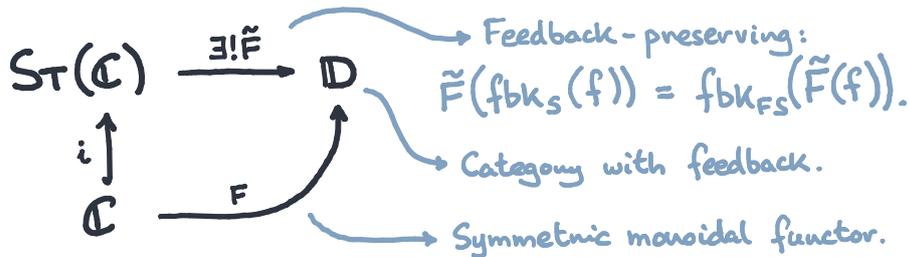


Categories with feedback

PROPOSITION. Let \mathbb{C} be a symmetric monoidal category. $ST(\mathbb{C})$ has a feedback structure given by

$$\text{fbk}_T \left(S \mid \begin{array}{c} S \\ T \\ A \end{array} \begin{array}{|c} \hline f \\ \hline \end{array} \begin{array}{c} S \\ T \\ B \end{array} \right) = \left(S \otimes T \mid \begin{array}{c} S \\ T \\ A \end{array} \begin{array}{|c} \hline f \\ \hline \end{array} \begin{array}{c} S \\ T \\ B \end{array} \right).$$

THEOREM. Let \mathbb{C} be a symmetric monoidal category. The symmetric monoidal category $ST(\mathbb{C})$ is the free category with feedback over \mathbb{C} , meaning that



Feedback, trace, and fixed-point semantics. Katis, Sabadini, Walters.

Categories with feedback

THEOREM. The following is an isomorphism of categories.

$$\text{SPAN}(\text{GRAPH})_* \cong \text{ST}(\text{SPAN}(\text{SET}))$$

Stateful ← ← Synchronization

$\text{SPAN}(\text{GRAPH})_*$ is the free category with feedback over $\text{SPAN}(\text{SET})$.

Categories with feedback

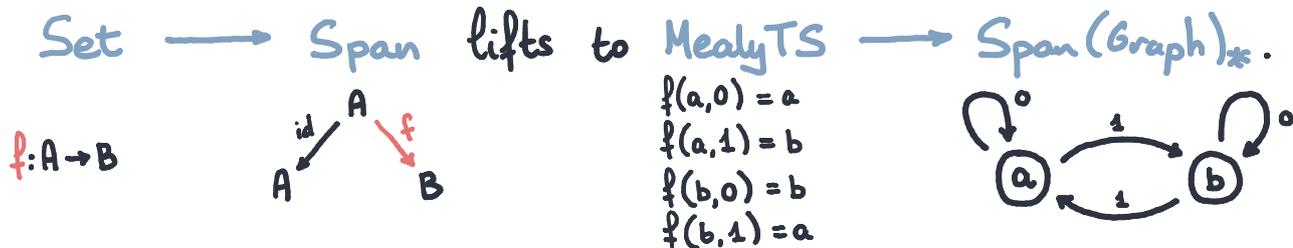
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Example:



Categories with feedback

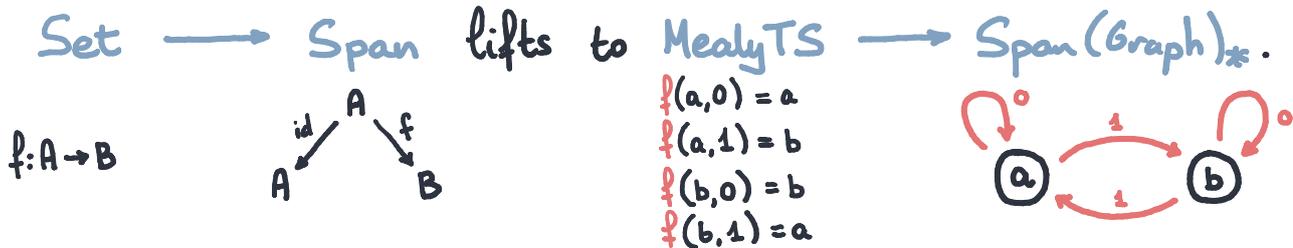
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Example:



PART 4:

GENERALIZING $\text{St}(\cdot)$

Generalizing Feedback

Feedback describes a particular flow of information.



normal flow



flow with feedback

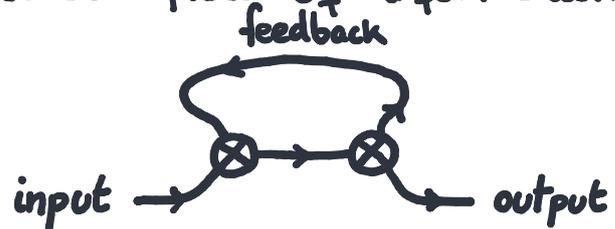
Open Diagrams via Coend Calculus, Mario Román, ACT'20

Generalizing Feedback

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normal flow



flow with feedback

These are monads in the bicategory PROF of profunctors:

$$\text{hom}(\mathbb{I}, \mathbb{O})$$

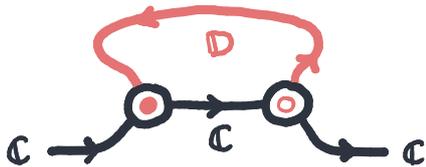
$$\int^{\text{SEC}} \text{hom}(S \otimes \mathbb{I}, S \otimes \mathbb{O})$$

monads correspond to a new assignment of morphisms to a category.

Open Diagrams via Coend Calculus, Mario Román, ACT'20

Generalizing Feedback

What is the most general form of feedback?



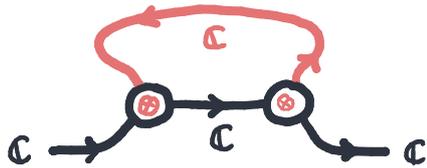
$$\text{St}_D(A, B) := \int^{D \in \mathcal{D}} \text{hom}(D \circ A, D \circ B)$$

The normal form theorem holds for any pair of monoidal actions (\circ, \circ) . These generalize:

- Traced categories without yanking.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.

Generalizing Feedback

What is the most general form of feedback?



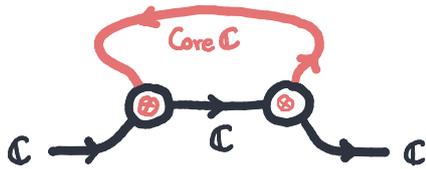
$$\text{St}_C(A, B) := \int^{S \in C} \text{hom}(S \otimes A, S \otimes B)$$

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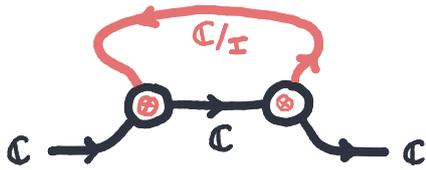
$$\text{St}_{\text{Core } \mathbb{C}}(A, B) := \int^{S \in \text{Core } \mathbb{C}} \text{hom}(S \otimes A, S \otimes B)$$

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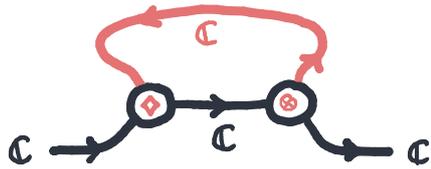
$$\text{St}_{C/I}(A, B) := \int^{S, s \in C/I} \text{hom}(S \otimes A, S \otimes B)$$

The normal form theorem holds for any pair of monoidal actions (\circ, \circ) . These generalize:

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Generalizing Feedback

What is the most general form of feedback?



$$\text{St}_{C, \diamond}(A, B) := \int^{S \in C} \text{hom}(\diamond S \otimes A, S \otimes B)$$

The normal form theorem holds for any pair of monoidal actions (\circ, \circ) . These generalize:

- Right / Left traced categories.
- Categories with feedback.
- Categories with initialized feedback.
- Delayed traces.

Conclusion

- $\text{SPAN}(\text{GRAPH})_* \cong \text{ST}(\text{SPAN}(\text{SET}))$.
- $\text{ST}(\cdot)$ commonly appears across the literature.
- $\text{ST}(\cdot)$ is the free category with feedback.
- Categories with feedback are a weakening of traces.
- Categories with feedback have a normal form theorem.
- $\text{ST}(\cdot)$ can be generalized to variants of feedback.
- Relate to the coalgebraic approach.

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