Diagrams for Effectful Categories
(& Premonoidal)

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Premonoidal Categories

DEFINITION (Power, Robinson). A binoidal category $(\mathbb{C} \otimes I)$, is a category with an object $I \in \mathbb{C}$ and an assignment on objects that is separately functorial on each component, $(A \otimes \bullet) : \mathbb{C} \to \mathbb{C}$ and $(\bullet \otimes B) : \mathbb{C} \to \mathbb{C}$.

- That is, $(\otimes)$ is a sesquifunctor.
- A morphism $f : A \to B$ is <u>central</u> if $(f \otimes id) ⨾ (id \otimes g) = (id \otimes g) ⨾ (f \otimes id)$ for any $g : A' \to B'$.

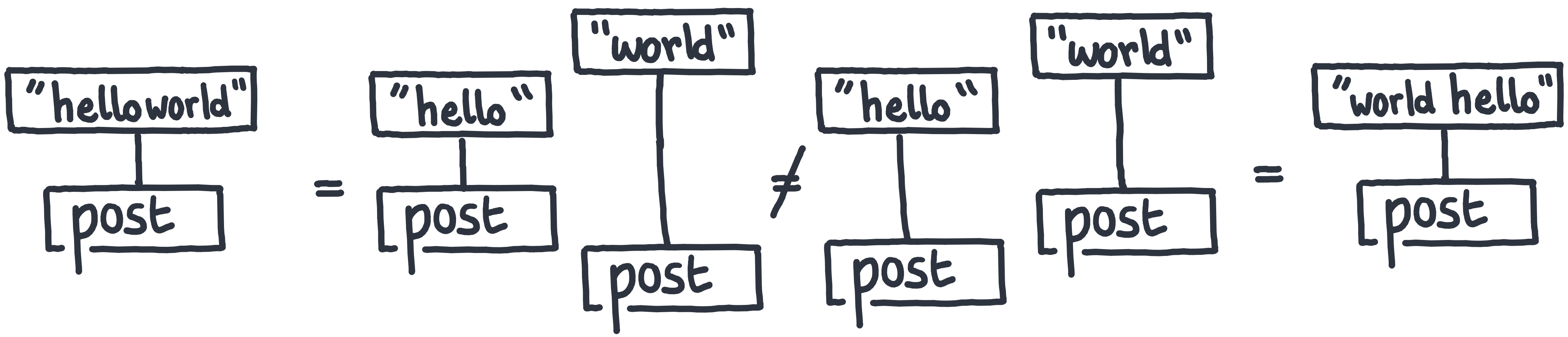
DEFINITION (Power, Robinson). A premonoidal category $(\mathbb{C}, \otimes, I)$ is a binoidal category with natural central isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad \rho_A : A \otimes I \to A, \quad \lambda_A : I \otimes A \to A,$$

satisfying pentagon and triangle equations. It is strict if these are identities.

PREMONOIDAL CATEGORIES

Premonoidal categories are monoidal categories without the interchange law.



PROPOSITION. The free algebras of a strong monad form a premonoidal category.
EXAMPLES. Processes with global state, e.g. $(\Sigma^* \times \bullet) : \mathrm{SET} \to \mathrm{SET}$.

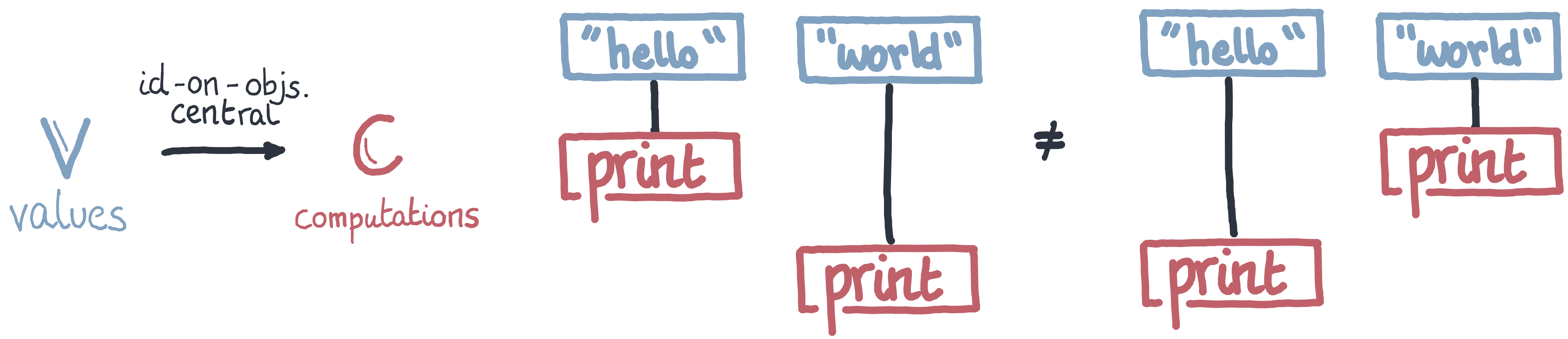
Power, Robinson, 97 ; Power, Thielecke, 99.

Effectful Categories

PROBLEM. There is not a good notion of premonoidal functor. We need premonoidals "with a chosen center".

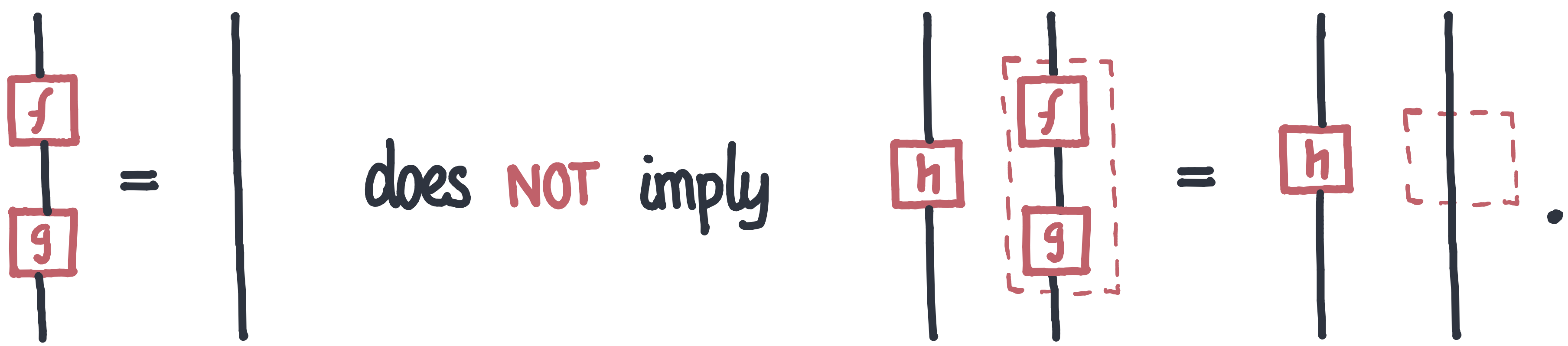
- This is the motivation for Freyd categories. Staton, Levy

DEFINITION. An *effectful category* is an identity-on-objects functor from a symmetric monoidal category $\mathbb{V}$ ("the values") to a symmetric premonoidal category $\mathbb{C}$ ("the computations"), strictly preserving the premonoidal structure and centrality.



Effectful Categories

Even when every effectful can be strictified, and even if we keep track of central and non-central morphisms,

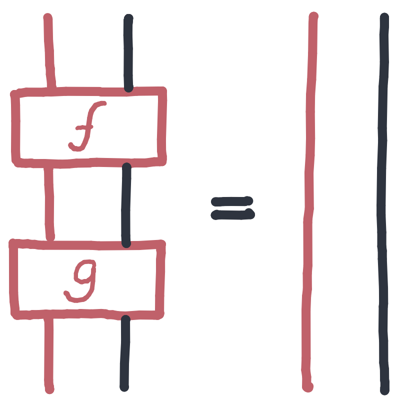


we lose 'locality' of transformations.

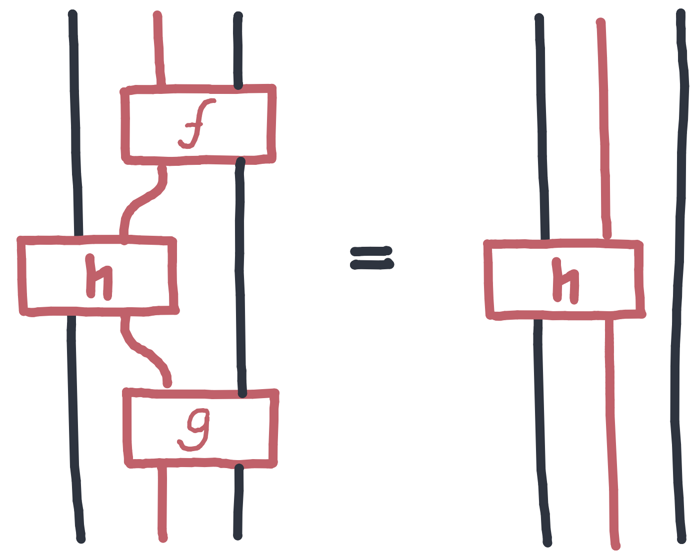
- Because of this discrepancy, the categories in ACT are seldom premonoidal. Diagrams are bad. What if there were a solution?

Effectful Categories

A solution was proposed in the 90s: add an extra wire. Jeffrey, 97.



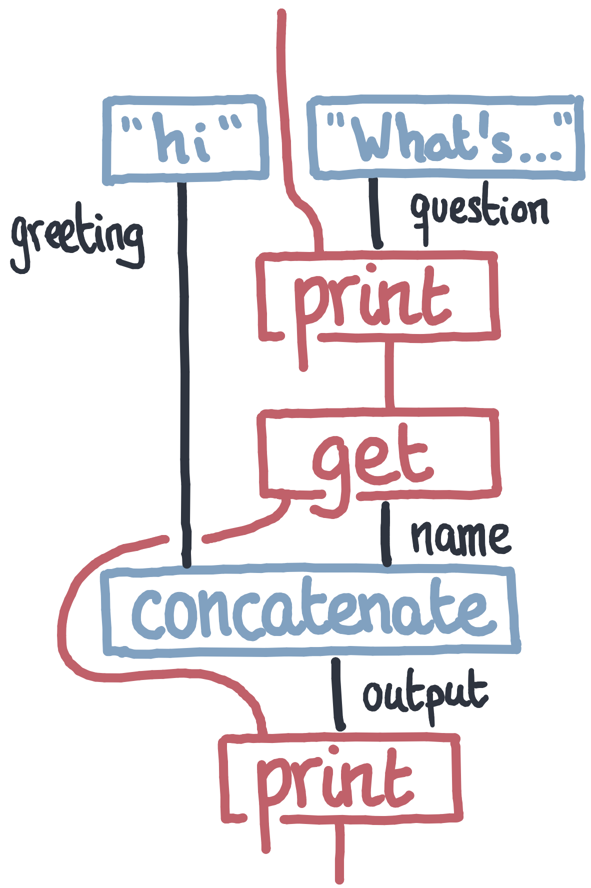
does NOT imply



Interpretation: Runtime, needed for computation, is a resource of your resource theory.

RUNTIME

Jeffrey, 97
Power, Thielecke, 99
Staton, Levy, 13



- An extra wire represents runtime: global state of the system and control over it.
- JEFFREY'S runtime is flexible: we can choose a class of pure morphisms. It works for effectful categories.

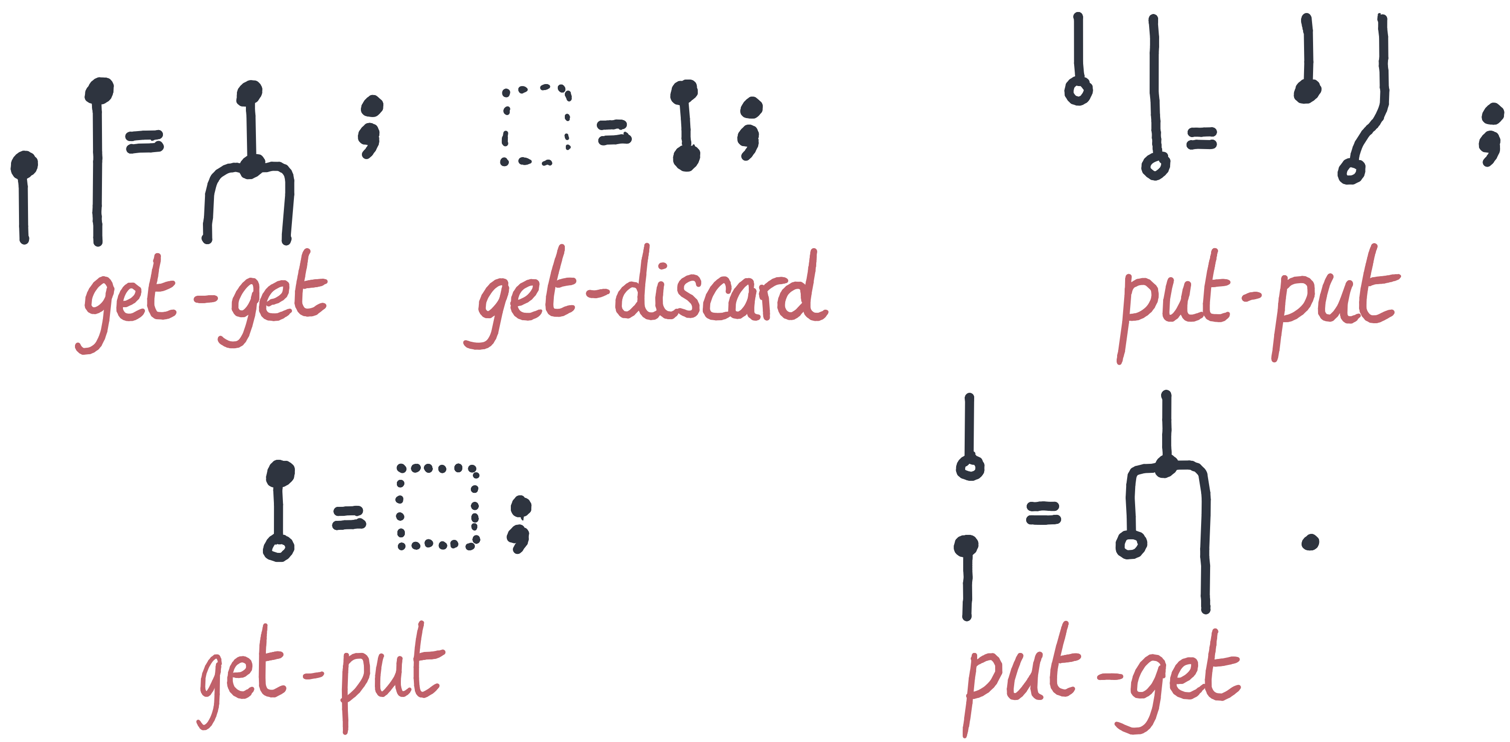
$$\underset{\text{values}}{\mathbb{V}} \xrightarrow[\text{central}]{\text{id-on-objs.}} \underset{\text{computations}}{\mathbb{C}}$$

- We can always choose $\mathcal{Z}\mathbb{C} \to \mathbb{C}$ for premonoidals.

From now on, without loss of generality, we work with effectful categories.

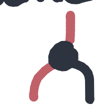
Why Effectful Categories (get/put example)

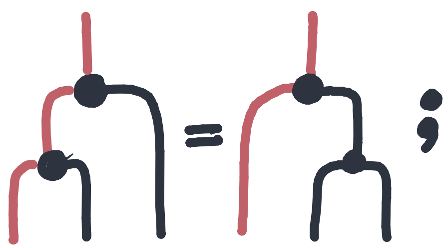
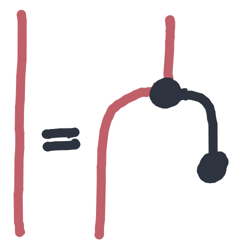
Many structures are clear once the runtime is present: consider the state monad $T_S X = S \to S \times X$ with get : $1 \to S$ and put : $S \to 1$.



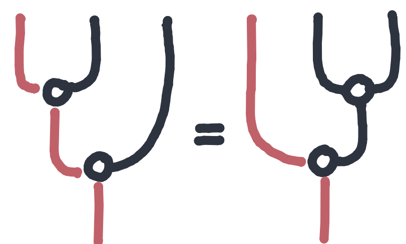
Consider some lens laws, drawn naïvely.

WHY EFFECTFUL CATEGORIES (GET/PUT EXAMPLE)

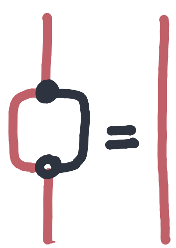
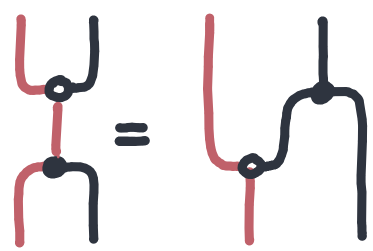
Many structures are clear once the runtime is present: consider the state monad $T_S X = S \to S \times X$ with get  $: 1 \to S$ and put  $: S \to 1$.

 ;  ;

comonoid (runtime coaction)

 with  ;

semimonoid (runtime semiaction)

 ;  .

special

semi-frobenius action

We can study a two-colour PROP instead.

CONTRIBUTIONS

1. THEOREM. The free effectful category in some generators is the free monoidal category over the same generators endowed with runtime.

$$\mathrm{PolyGraph} \underset{\text{Forget}}{\overset{\text{String}}{\rightleftarrows}} \mathrm{StrictEffectful} \qquad (\bot)$$

(String Diagrams)

2. THEOREM. Effectful categories are pseudomonoids in a monoidal bicategory of promonads; in the same way that monoidal categories are pseudomonoids in a monoidal bicategory of categories.

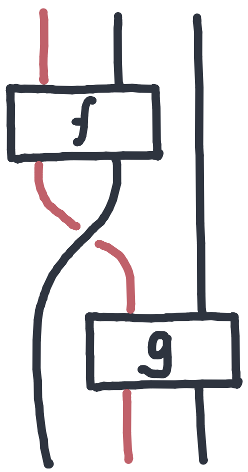
Pseudomonoid →

	Categories	Monoidal categories	
Effects ↓ (Arrows)	Id-on-objects	Effectful categories	(Freyd categories)

PART 1: RUNTIME

Adding Runtime

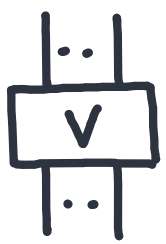
Jeffrey's Notation. Avoid interchange by adding an extra wire: the runtime.



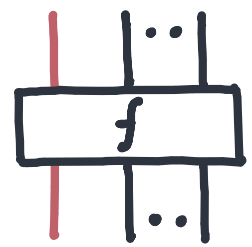
Theorem. The morphisms $A_0 \otimes \cdots \otimes A_n \to B_0 \otimes \cdots \otimes B_m$ of the free effectful category over a 'signature' $\mathcal{H} \to \mathcal{G}$ are morphisms

$$R \otimes A_0 \otimes \cdots \otimes A_n \to R \otimes B_0 \otimes \cdots \otimes B_m$$

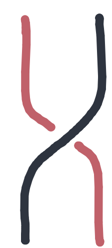
of a monoidal category with generators



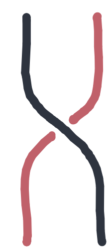
for $v \in \mathcal{V}$;



for $f \in \mathcal{G}$;



;



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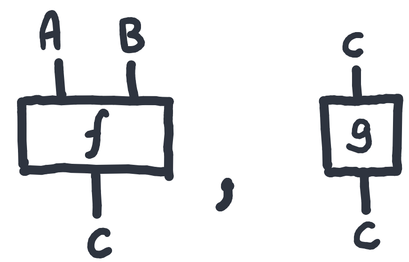
and the expected axioms.

POLYGRAPHS

DEFINITION. A polygraph $\mathcal{G}$ is given by a set of objects, $\mathcal{G}_{obj}$, and a set of arrows $\mathcal{G}(A_0, \ldots, A_n; B_0, \ldots B_n)$ for sequences $A_0, \ldots, A_n$ and $B_0, \ldots, B_n$.

EXAMPLE.

$\mathcal{G} = \{ A, B, C \mid$



$\}$

Polygraphs give signatures for monoidal categories: string diagrams over a polygraph form the free strict monoidal category on it.

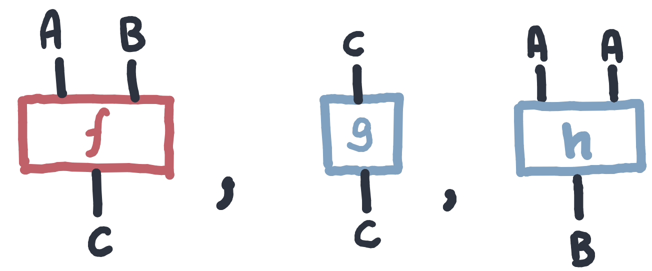
THEOREM. String diagrams construct an adjunction

$$\textsc{Polygraph} \underset{\text{Forget}}{\overset{\text{Strings}}{\rightleftarrows}} \textsc{StrictMonCat}, \quad \text{Strings} \dashv \text{Forget}.$$

PolyGraphs

DEFINITION. A polygraph couple $(\mathcal{H}, \mathcal{G})$ is a pair of polygraphs sharing the same objects, $\mathcal{H}_{obj} = \mathcal{G}_{obj}$.

EXAMPLE.

$(\mathcal{H}, \mathcal{G}) = \{ A, B, C \mid$ 

$\}$

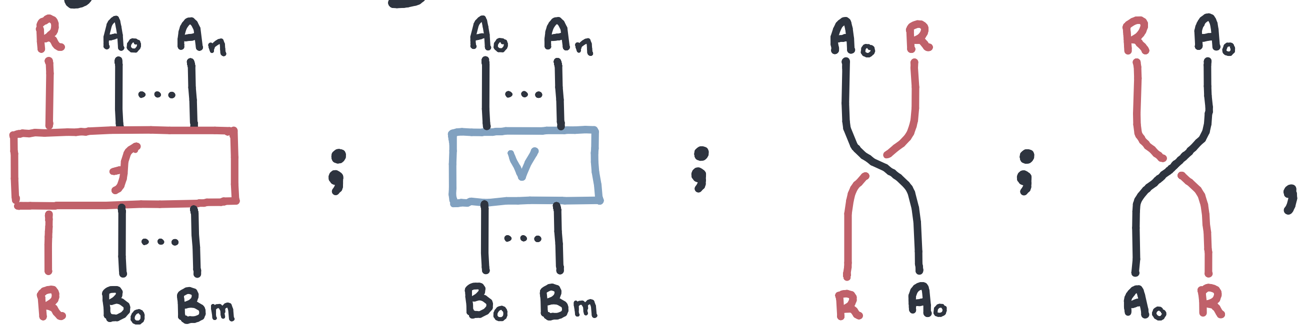
Polygraph couples give signatures for effectful categories.

THEOREM. String diagrams with runtime construct an adjunction

$$\textsc{PolygraphCouple} \underset{\text{Forget}}{\overset{\text{Strings}}{\rightleftarrows}} \textsc{StrictEffectful}, \quad \text{Strings} \dashv \text{Forget}.$$

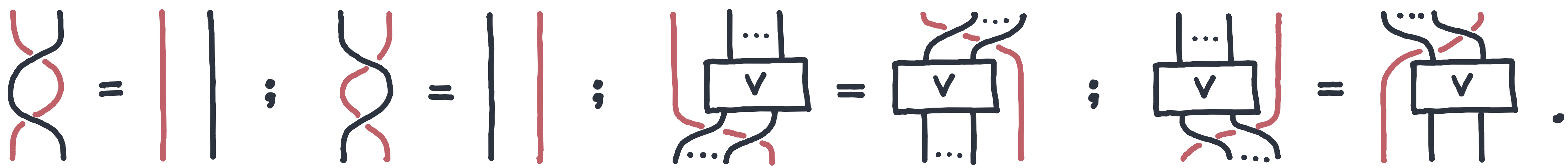
String Diagrams with Runtime

String diagrams generated by



for $f \in \mathcal{G}(A_0, \dots, A_n ; B_0, \dots, B_m)$, and $v \in \mathcal{H}(A_0, \dots, A_n ; B_0, \dots, B_m)$.

and quotiented by braiding axioms, asking R to be on the Drinfeld centre:



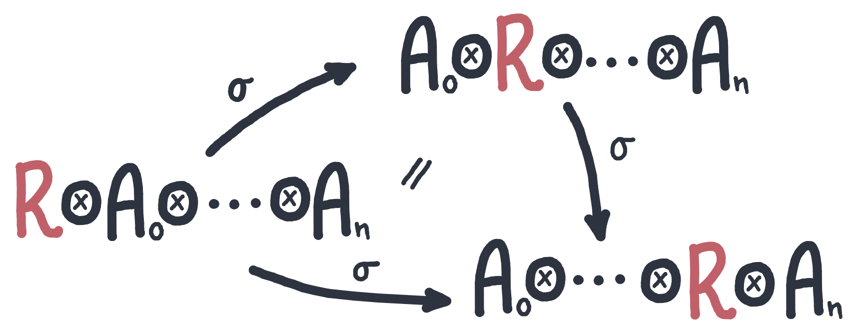
We prove that diagrams $R \otimes A_0 \otimes \cdots \otimes A_n \to R \otimes B_0 \otimes \cdots \otimes B_m$ form an effectful category.

Cliques

Technical problem: $R \otimes A_0 \otimes \cdots \otimes A_n \to R \otimes B_0 \otimes \cdots \otimes B_m$ always assumes that the runtime is on the left.

DEFINITION. A clique is a collection of objects, $\{A_i\}_{i \in I}$, together with an isomorphism $\gamma_{ij} : A_i \to A_j$, such that $\gamma_{ii} = id_i$, and $\gamma_{ij} \mathbin{;} \gamma_{jk} = \gamma_{ik}$.

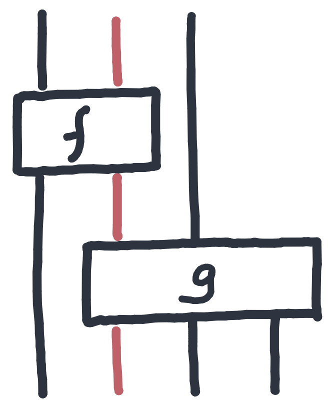
CONSTRUCTION. The braid clique on a list of objects $[A_0, \dots, A_n]$ has R inserted at each position.



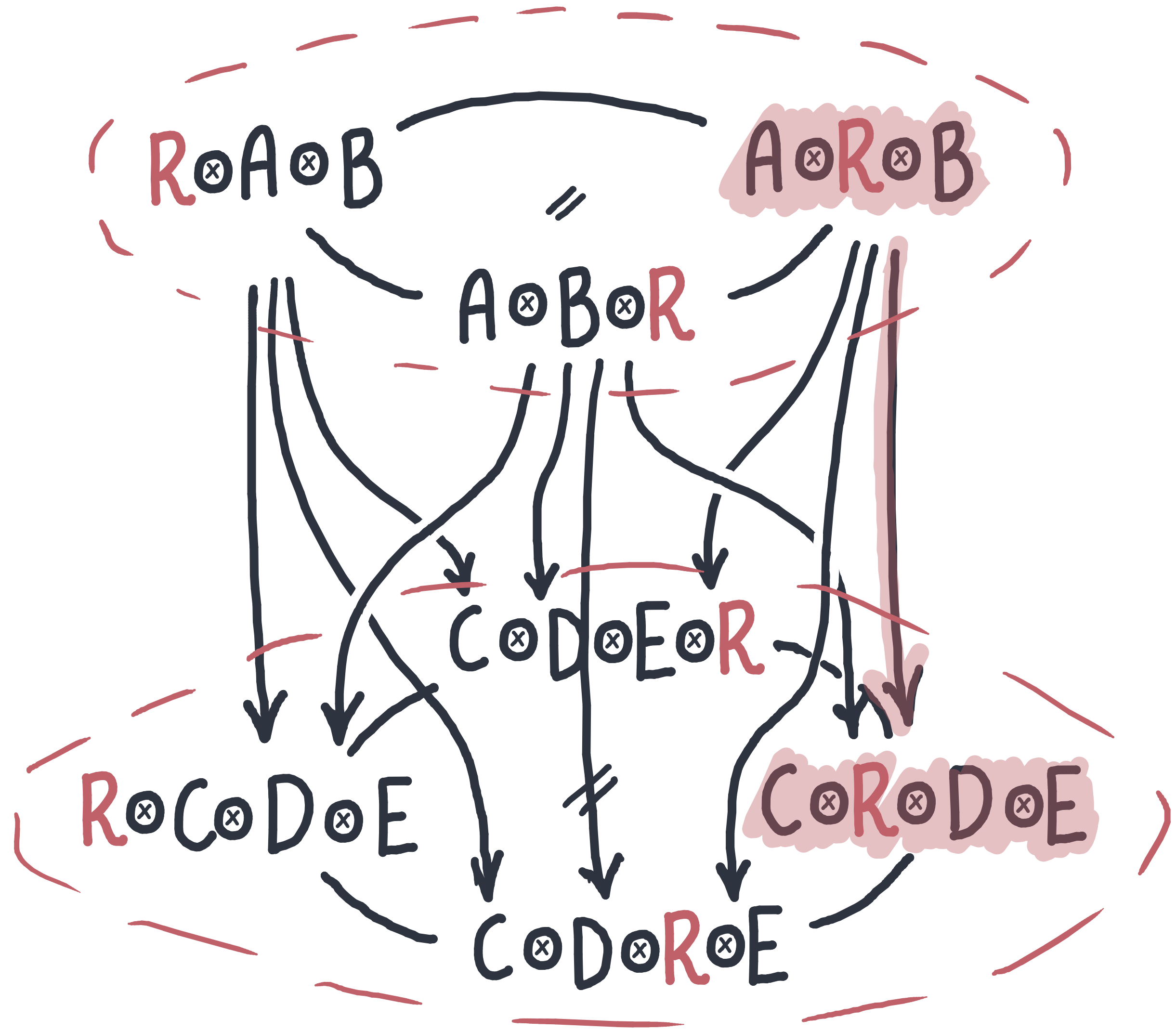
~ See Trimble's intuition of monoidal coherence, on the nLab.

Cliques

$A \otimes R \otimes B$

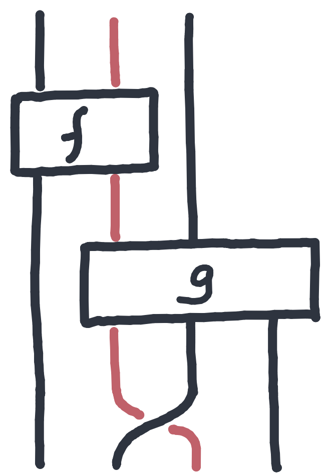


$C \otimes R \otimes D \otimes E$

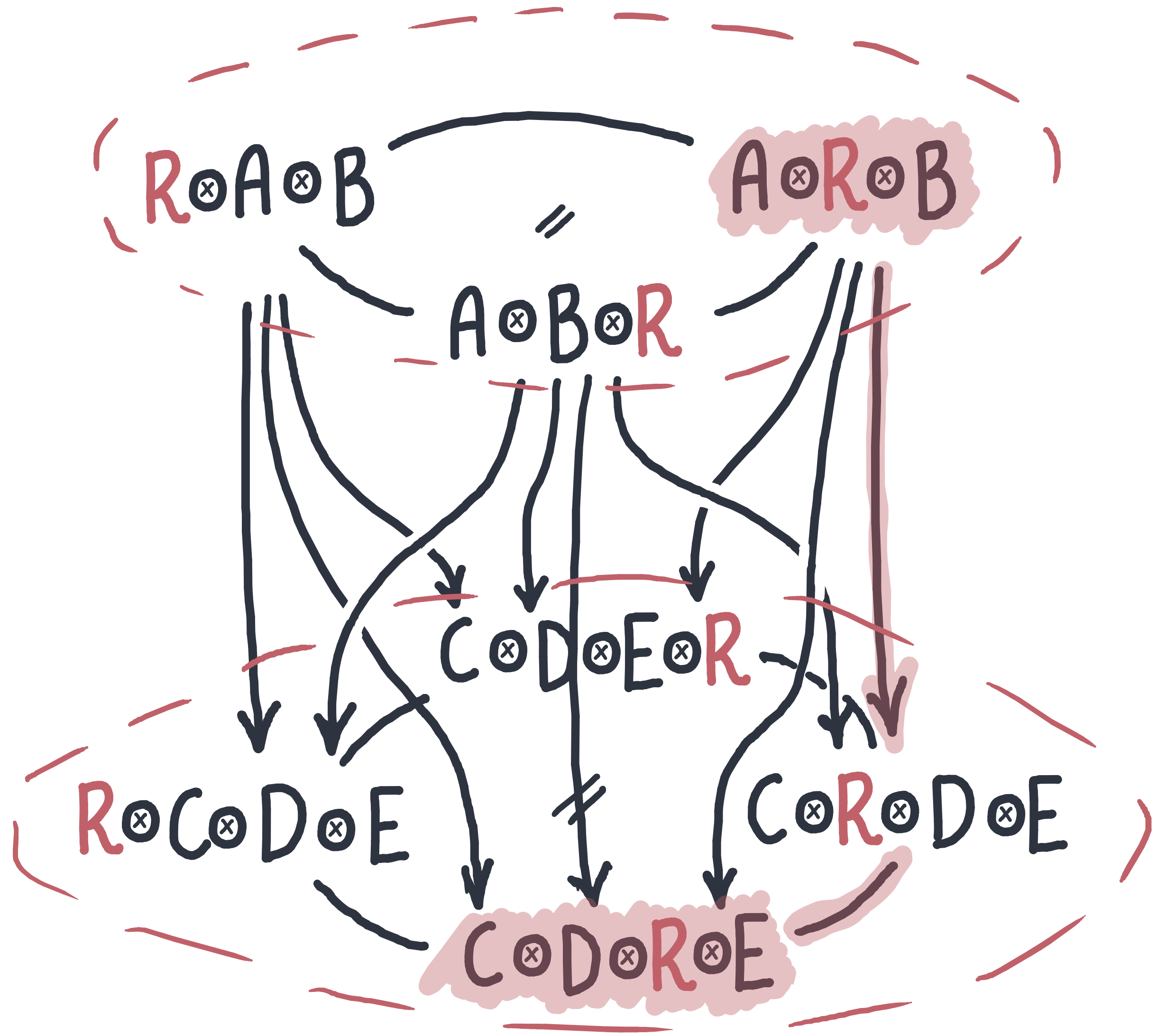


Cliques

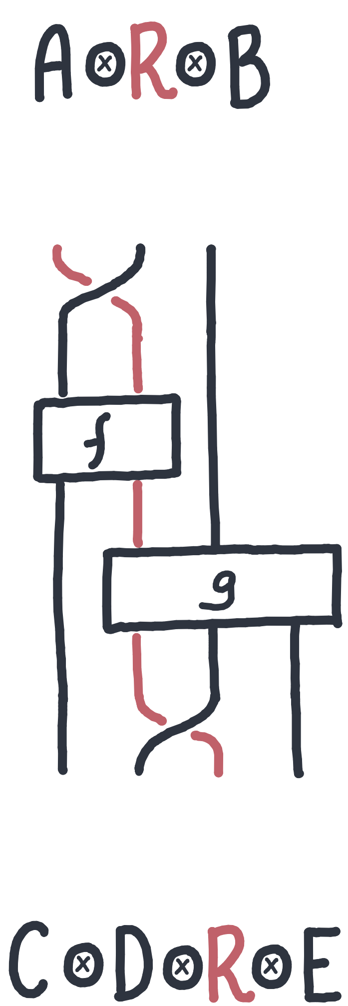
$A \otimes R \otimes B$

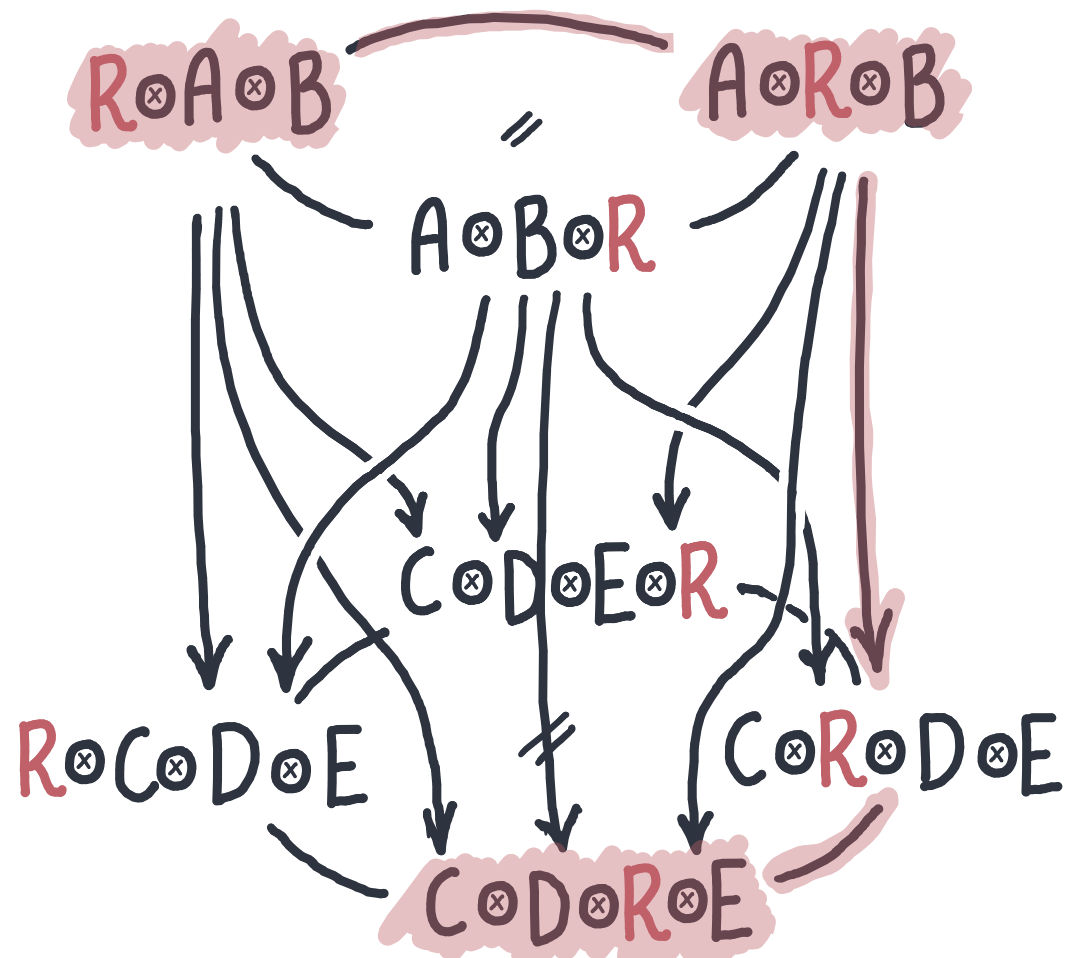


$C \otimes D \otimes R \otimes E$



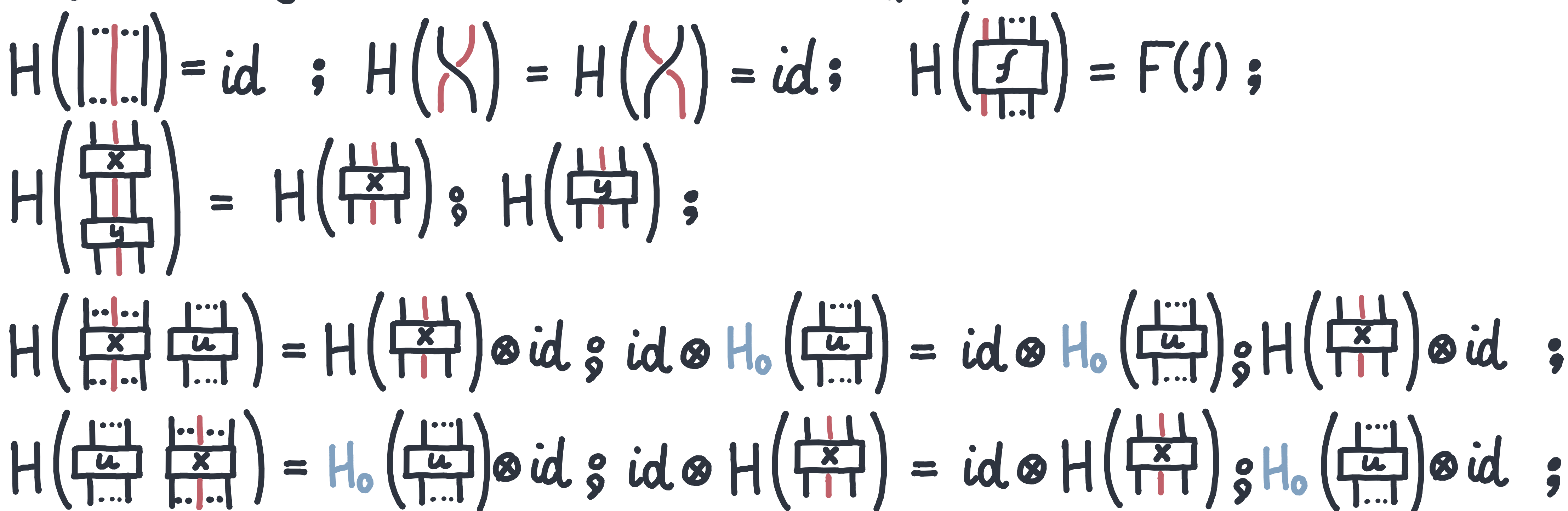
Cliques





String Diagrams with Runtime

Finally, the assignment determining the universal effectful functor is



THEOREM. String diagrams with runtime construct an adjunction

$$\mathbf{PolygraphCouple} \underset{\text{Forget}}{\overset{\text{Strings}}{\rightleftarrows}} \mathbf{StrictEffectful}. \quad (\bot)$$

PART 2: PROMONADS

PROFUNCTORS

A **profunctor** $P : \mathbb{C}^{op} \to \mathbb{D}$ is the same thing as a functor $P : \mathbb{C}^{op} \times \mathbb{D} \to \mathrm{SET}$.

Alternatively, a family of sets, $P(A,B)$, with actions

$(>) : \hom(A',A) \times P(A\ B) \to P(A',B)$ $\quad (>) : P(A,B) \times \hom(B,B') \to P(A,B')$ such that

$$f_1 ⨟ f_2 > p = f_1 > f_2 > p \qquad p < g_1 ⨟ g_2 = p < g_1 < g_2 \qquad (f > p) < g = f > (p < g)$$

$$id > p = p \qquad p < id = p$$

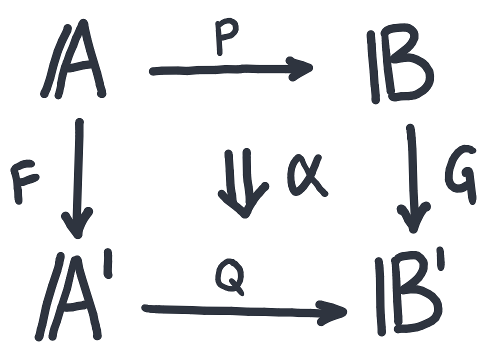
Profunctors compose as bimodules: $P \diamond Q\,(A,C)$ has elements (p,q) for $p \in P(A\ B)$ and $q \in Q(B,C)$, quotiented by $(p < g \,;\, q) \sim (p \,;\, g > q)$. This is a coend

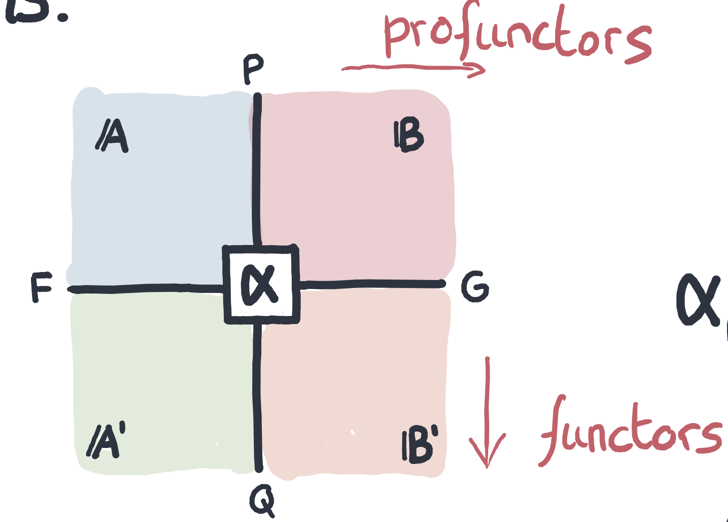
$$(P \diamond Q)(A,C) = \int^{B \in \mathbb{D}} P(A,B) \times Q(B,C).$$

Justesen Bunge Bénabou, 73

Double Category Cat

Consider the double category of categories, functors, profunctors and natural transformations.





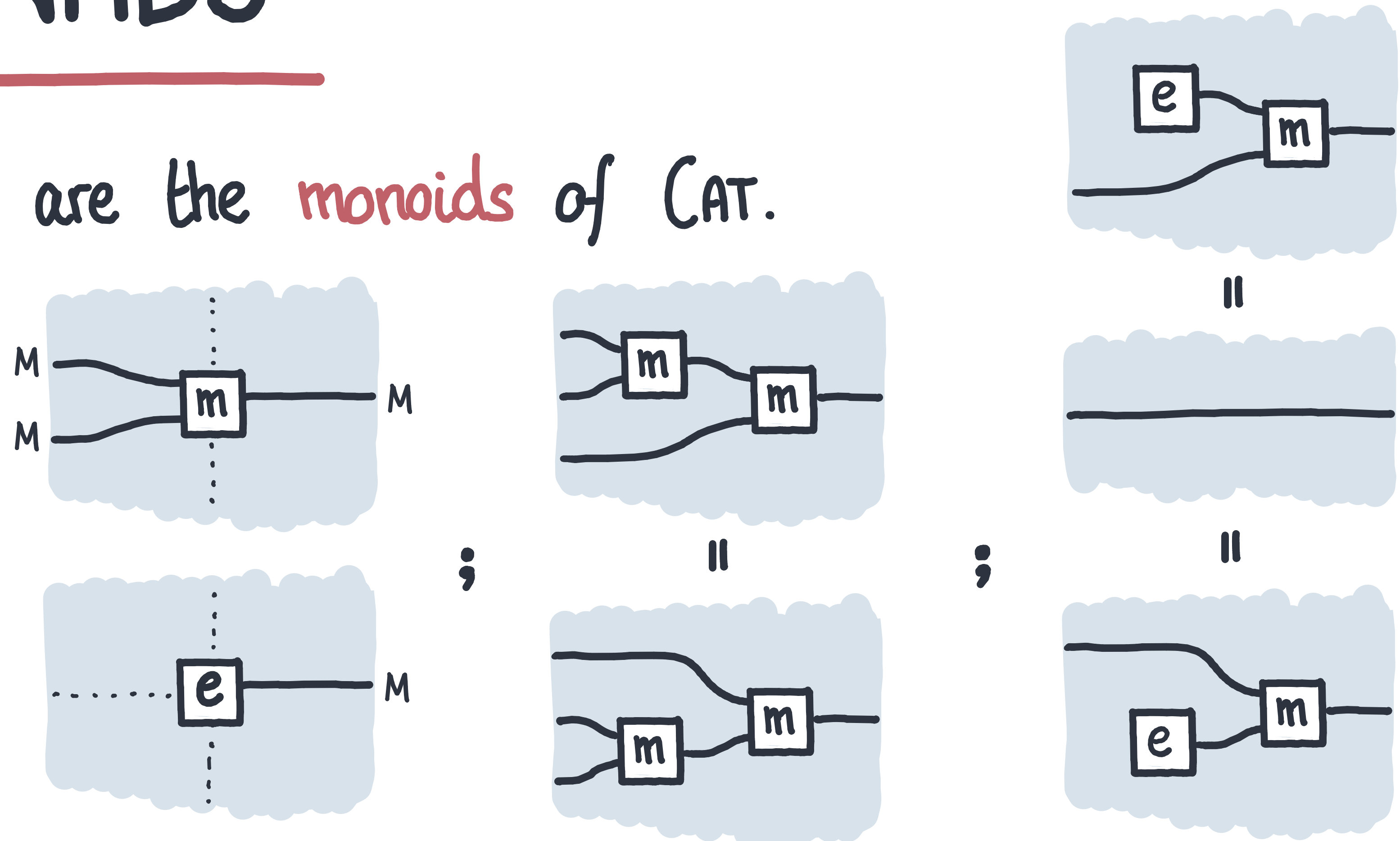
$$\alpha_{A,B} : P(A,B) \to Q(FA,GB)$$

Profunctors compose by **coends**, $P \diamond Q\ (A,C) = \int^{B \in \mathbb{B}} P(A,B) \times Q(B,C)$;
the identity is the **hom**-profunctor.

Ehresmann, 63 Grandis, Parè, 04 Myers 18

Monads

Monads are the monoids of Cat.



$I \Rightarrow M$

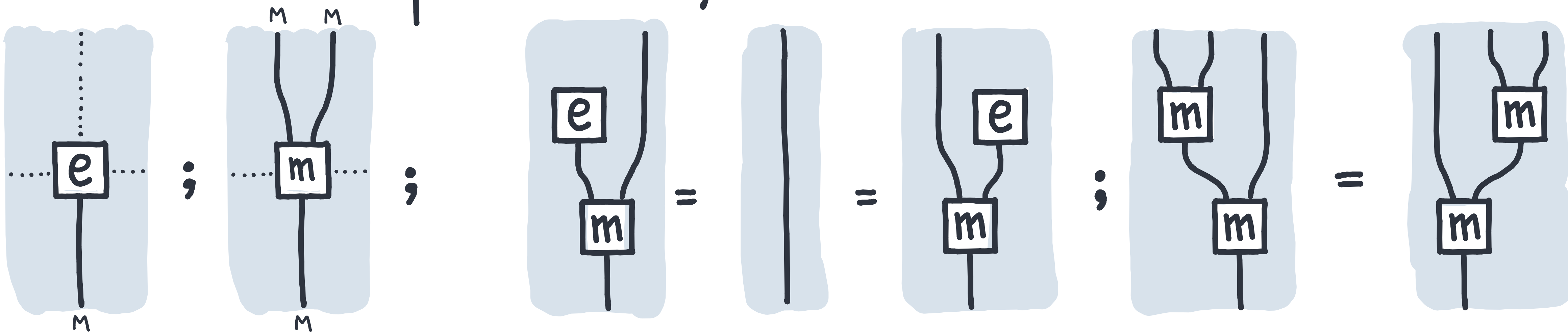
$M \circ M \Rightarrow M$

$\eta : X \to MX$

$\mu : MMX \to MX$

ProMonads

Promonads are the promonoids of Cat.



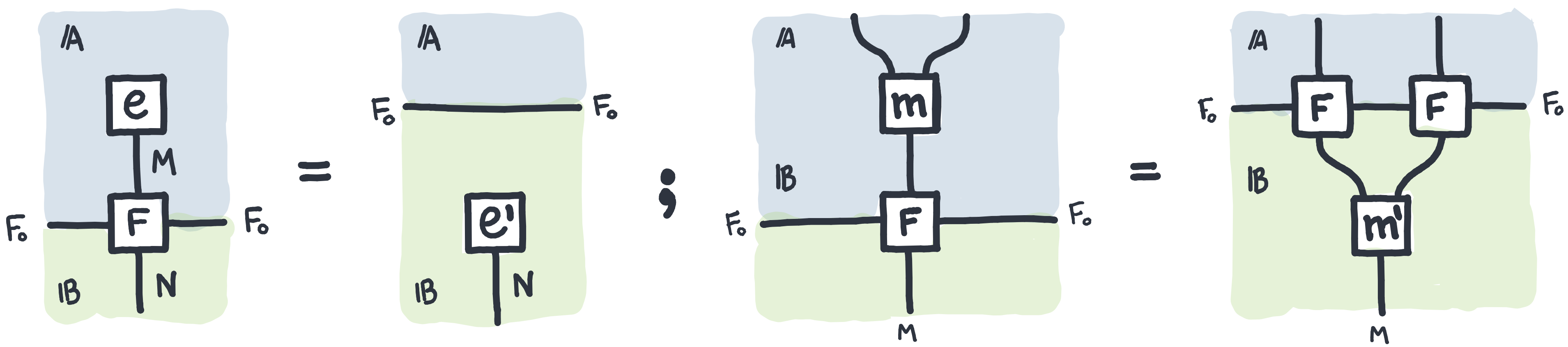
$\hom \Rightarrow M$ $\quad (\cdot)^\circ : \hom(A,B) \to M(A,B)$

$M \diamond M \Rightarrow M$ $\quad \star : \int^B M(A,B) \times M(B,C) \to M(A,C)$

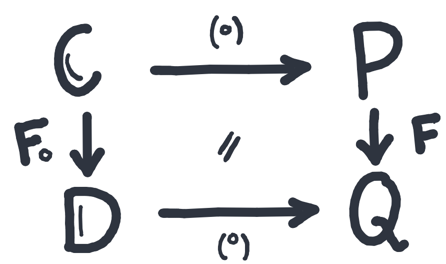
PROPOSITION. A promonad $P : \mathbb{C} \nrightarrow \mathbb{C}$ is exactly an id-on-objects functor $i : \mathbb{C} \to \tilde{P}$.

ProMonad Morphisms

Promonad homomorphisms are promonoid homomorphisms in the double category CAT.



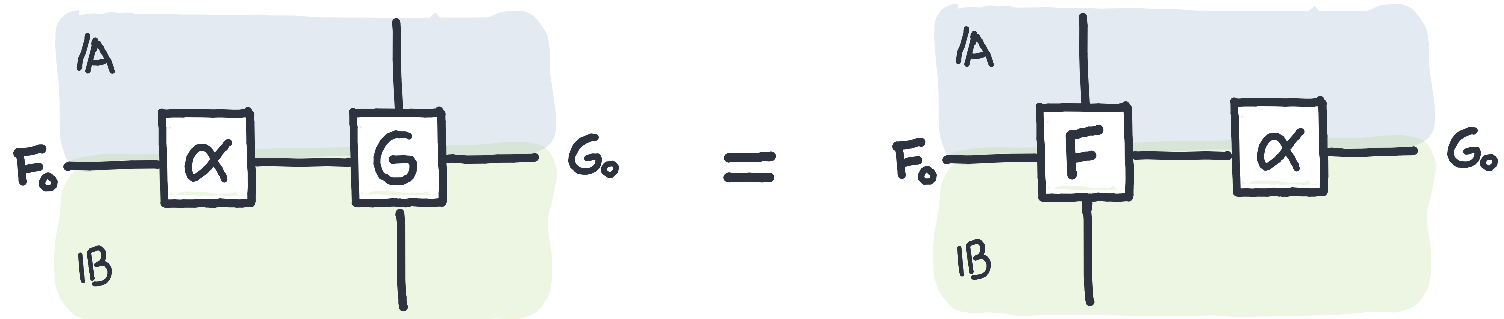
PROPOSITION. A promonad homomorphism is exactly a commuting square.



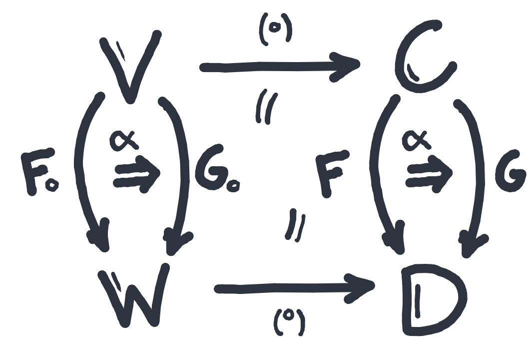
$$F(f \star g) = Ff \star Fg,$$
$$F(f^{\circ}) = F_{\circ}(f)^{\circ}.$$

ProMonad Transformation

Promonad transformations are promonoid transformations in the double category CAT.



PROPOSITION. A promonad transformation is exactly a cylinder transformation of squares.



Pure Tensor

We already have a bicategory: promonads, morphisms and transformations. There exists a notion of free product with commuting subgroups.

DEFINITION. The pure tensor of promonads, $\mathbb{C} \colon \mathbb{V} \nrightarrow \mathbb{V}$ and $\mathbb{D} \colon \mathbb{W} \nrightarrow \mathbb{W}$, is a promonad, $\mathbb{C} * \mathbb{D} \colon \mathbb{V} \times \mathbb{W} \nrightarrow \mathbb{V} \times \mathbb{W}$, whose elements are generated by

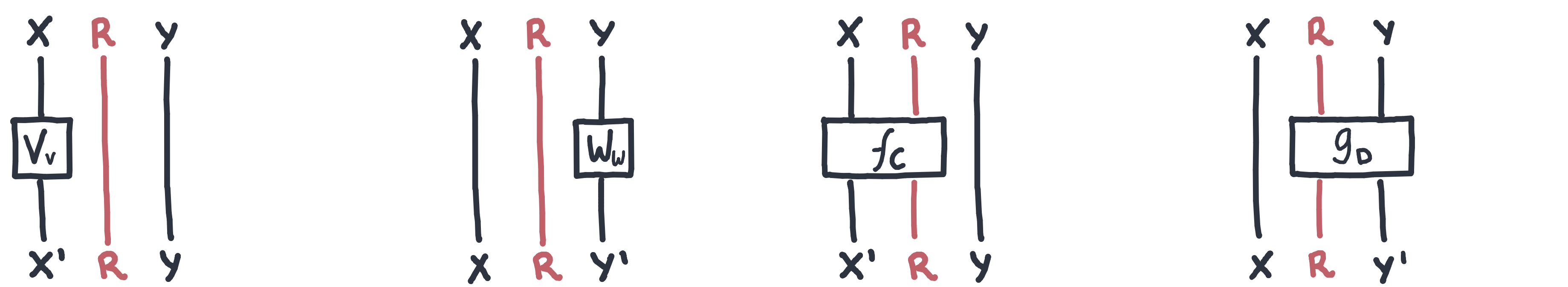
$p_{\mathbb{C}} \in \mathbb{C} * \mathbb{D}(X,Y;X',Y)$ for $p \in \mathbb{C}(X,X')$, or
$q_{\mathbb{D}} \in \mathbb{C} * \mathbb{D}(X,Y;X,Y')$ for $q \in \mathbb{D}(Y,Y')$.

quotiented by $f_{\mathbb{C}} ; w^{\circ}_{\mathbb{D}} = w^{\circ}_{\mathbb{D}} ; f_{\mathbb{C}}$ and $g_{\mathbb{D}} ; v^{\circ}_{\mathbb{C}} = v^{\circ}_{\mathbb{C}} ; g_{\mathbb{D}}$, plus other axioms.

THEOREM. There exist a pair of promonad homomorphisms
$L \colon \mathbb{C} \times \mathbb{W} \to \mathbb{C} * \mathbb{D}$ and $R \colon \mathbb{V} \times \mathbb{D} \to \mathbb{C} * \mathbb{D}$
and the pure tensor is universal with respect to them.

Pure Tensor

We already have a bicategory: promonads, morphisms and transformations.
There exists a notion of free product with commuting subgroups.



for $v \in \mathbb{V}(X,X')$ for $w \in \mathbb{W}(Y,Y')$ for $f \in \mathbb{C}(X,X')$ for $g \in \mathbb{D}(X,X')$

All these commutativity restrictions are better encoded by string diagrams.

- Thanks to an extra wire.

Effectfuls as Pseudomonoids

Pseudomonoids are 2-dimensional monoids.

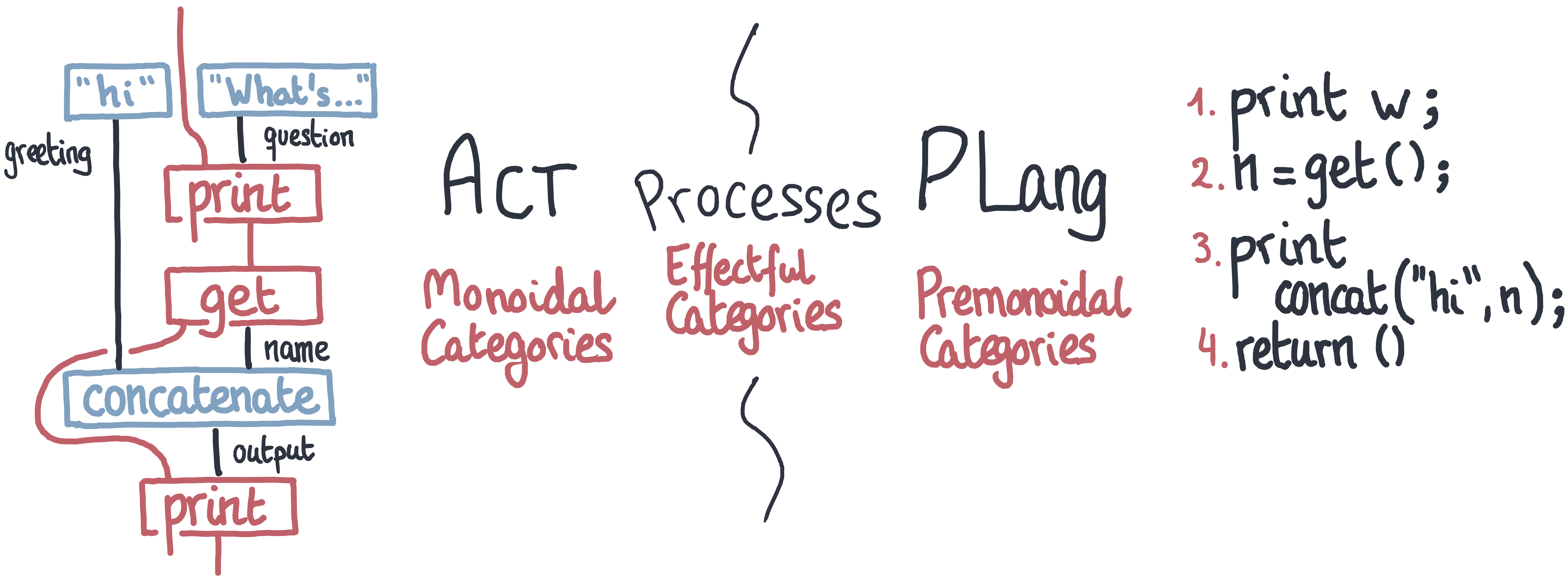
THEOREM (Street, Day). Monoidal categories are pseudomonoids in the monoidal bicategory of categories.

THEOREM. Effectful categories (and thus premonoidal categories with their centre) are pseudomonoids in the monoidal bicategory of promonads with the pure tensor.

PART 3: CONCLUSION

Conclusion

Premonoidal categories are already widely employed in programming languages. This is an invitation to use them in ACT, without leaving the comfort of monoidal categories and string diagrams.

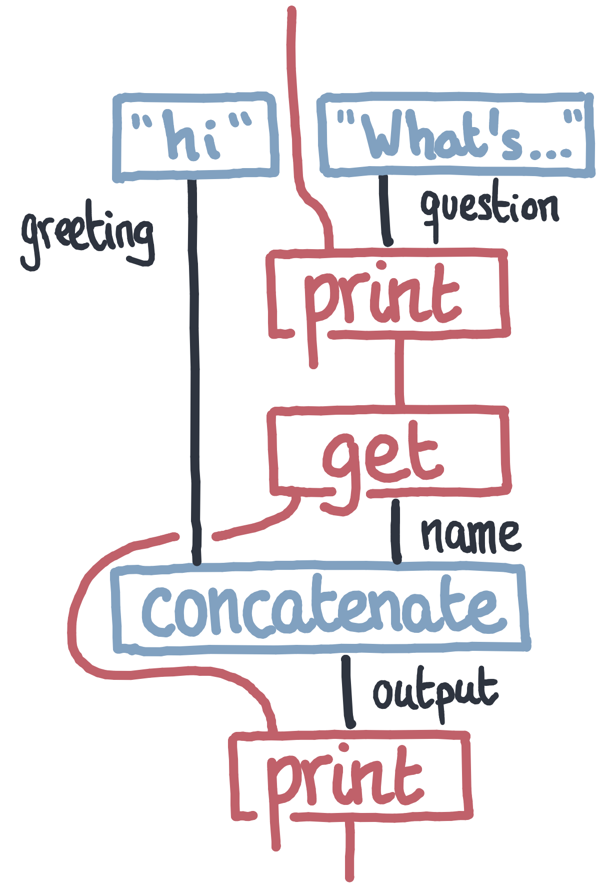


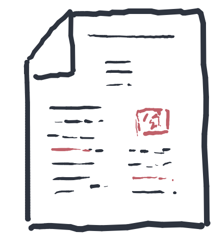
ARROW-DO NOTATION

Correspondence between string diagrams and the DO-notation.

EXAMPLE. HelloProgram

```
proc () -> do
   question <- "What's your name?"
   () <- print -< question
   name <- get -< ()
   () <- print -< concatenate("hi, ", name)
return ()
```





Promonads and String Diagrams for Effectful Categories (Román, 2022)

ArXiv: 2205.07664