Collages of String Diagrams

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Collages of String Diagrams

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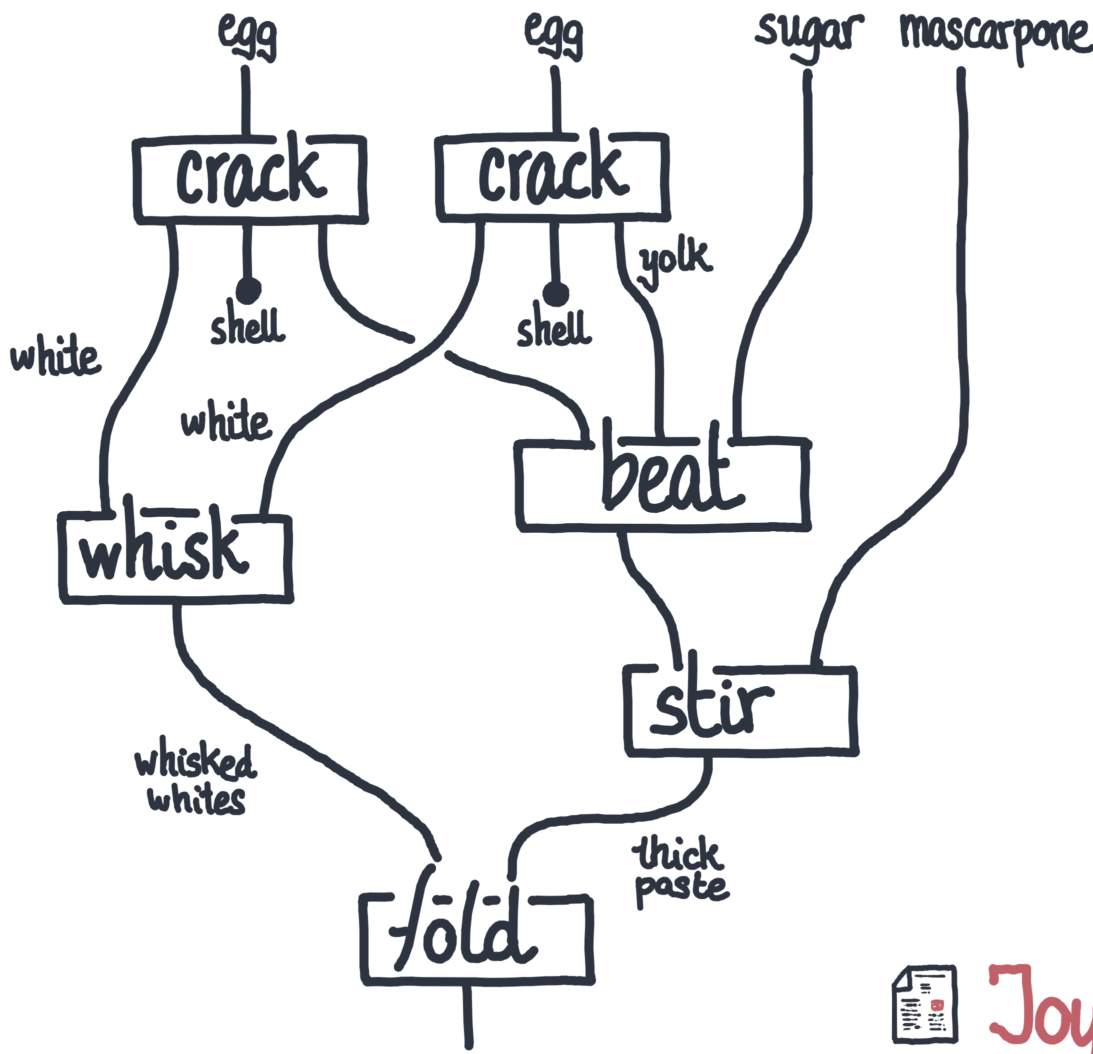
Mario Román

Collages of String Diagrams.

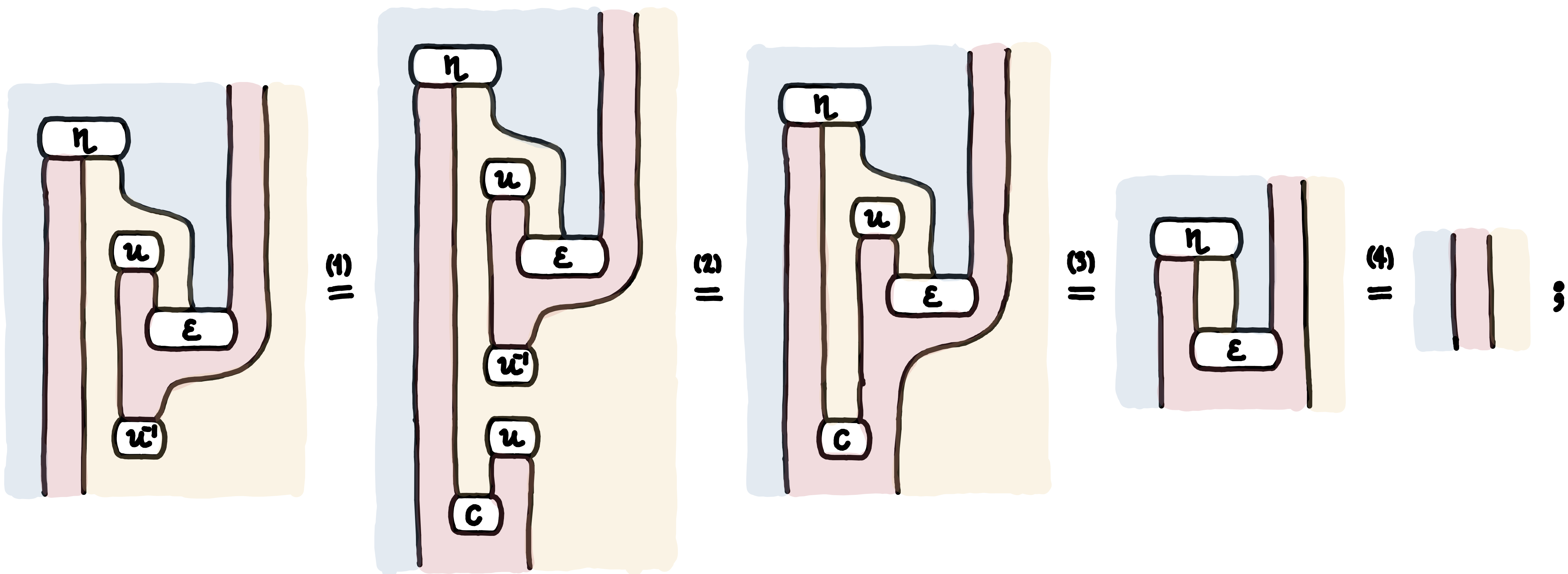
String Diagrams of Monoidals



Joyal, Street. Geom. of Tensor Calc.

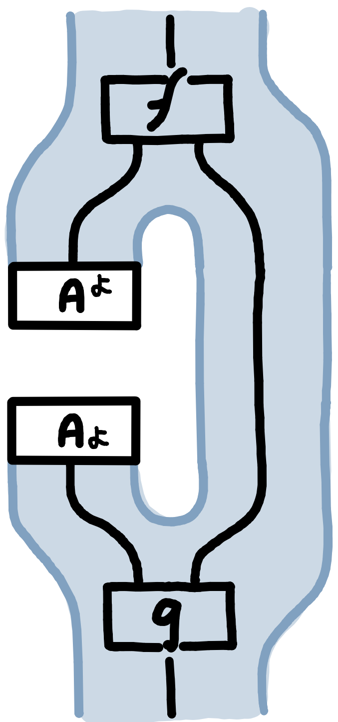
Sobociński. Graphical Linear Algebra.

String Diagrams of Bicategories

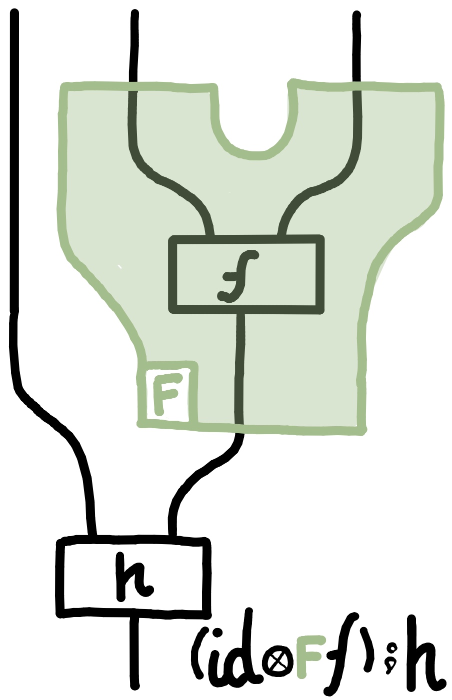


Marsden. Category theory with string diagrams.

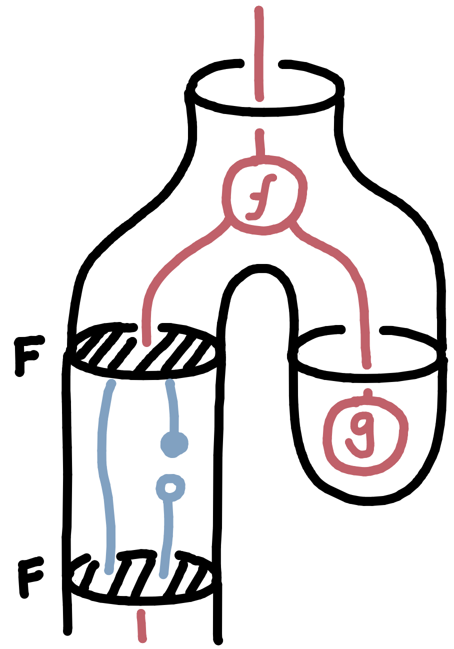
String Diagrams, Extended



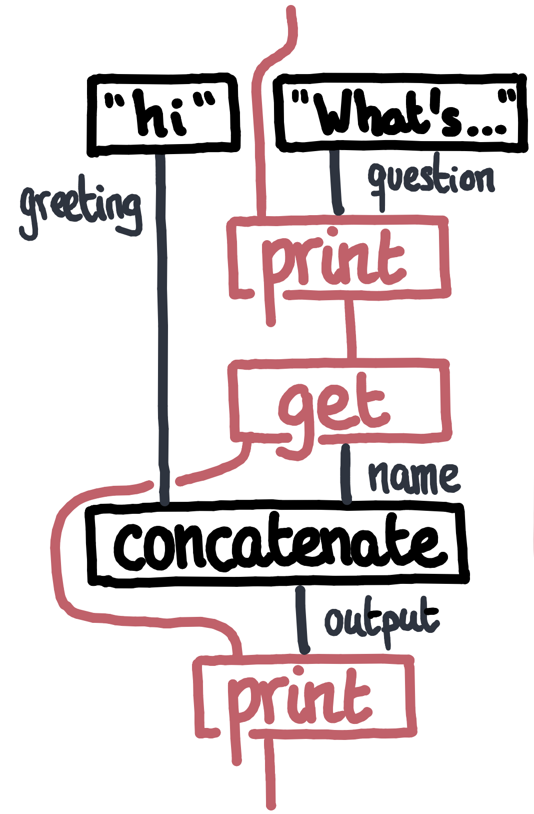
- Chiribella et al.
- Coecke et al.
- Hedges / Riley
- Kissinger, Uijlen



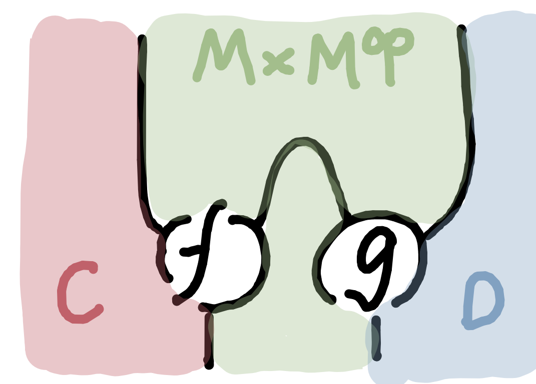
- Cockett, Pastro
- Melliès



- Bartlett, Douglas Schommer-Pries, Vicary
- Hu, Vicary
- Lobski, Zanasi



- Jeffrey
- Staton, Møgelberg
- Román



- Braithwaite

1. Bimodular Categories

Bimodular Categories

Monoid	Monoidal Category
Bimodule	Bimodular Category

Definition. A *bimodular category* ${}_{A}X_{B}$ is a category X with a pair of compatible actions

$$(\triangleright) : A \times X \to X \quad \text{and} \quad (\triangleleft) : X \times B \to X$$

with coherent natural isomorphisms

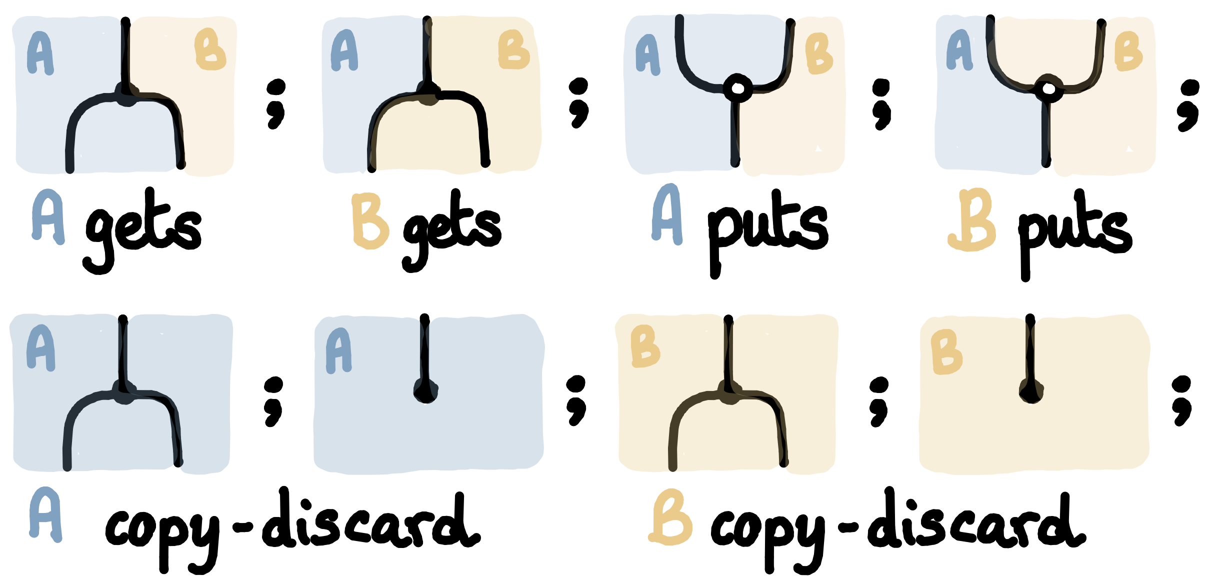
$$M_1 \triangleright M_2 \triangleright X \cong (M_1 \otimes M_2) \triangleright X; \qquad X \triangleleft N_1 \triangleleft N_2 \cong X \triangleleft (N_1 \otimes N_2);$$

$$I \triangleright X \cong X; \qquad X \triangleleft I \cong X;$$

$$(M \triangleright X) \triangleleft N \cong M \triangleright (X \triangleleft N).$$

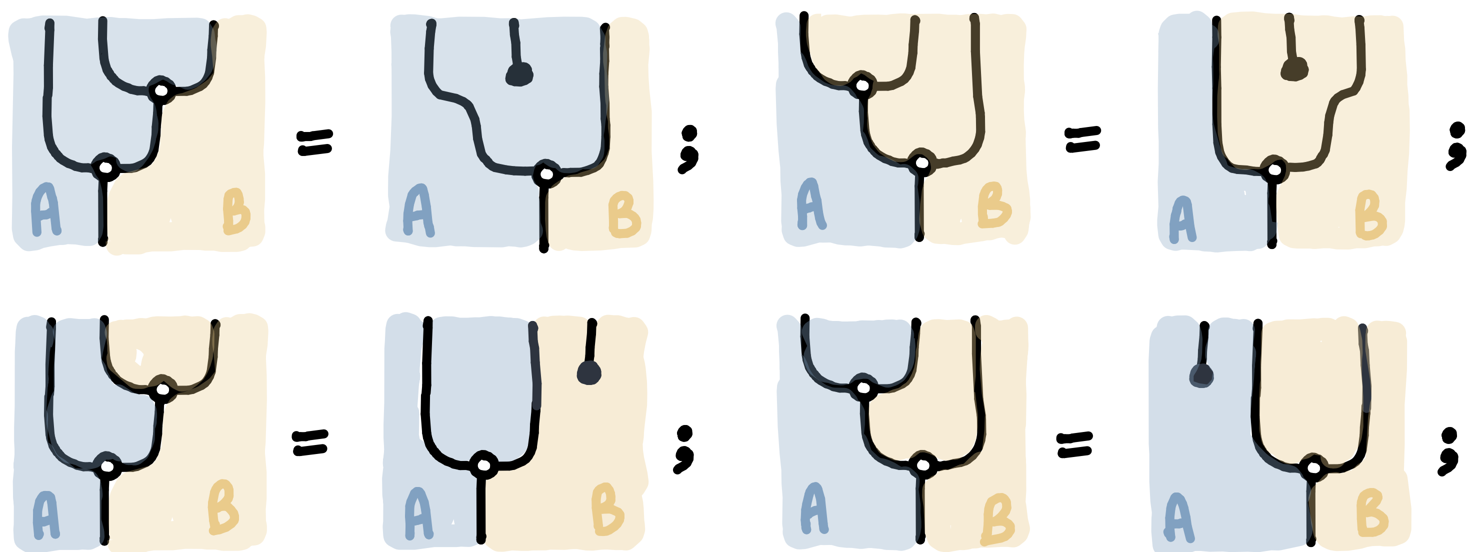
SHARED STATE

Two parties, A and B, share a common memory where they can read and write.

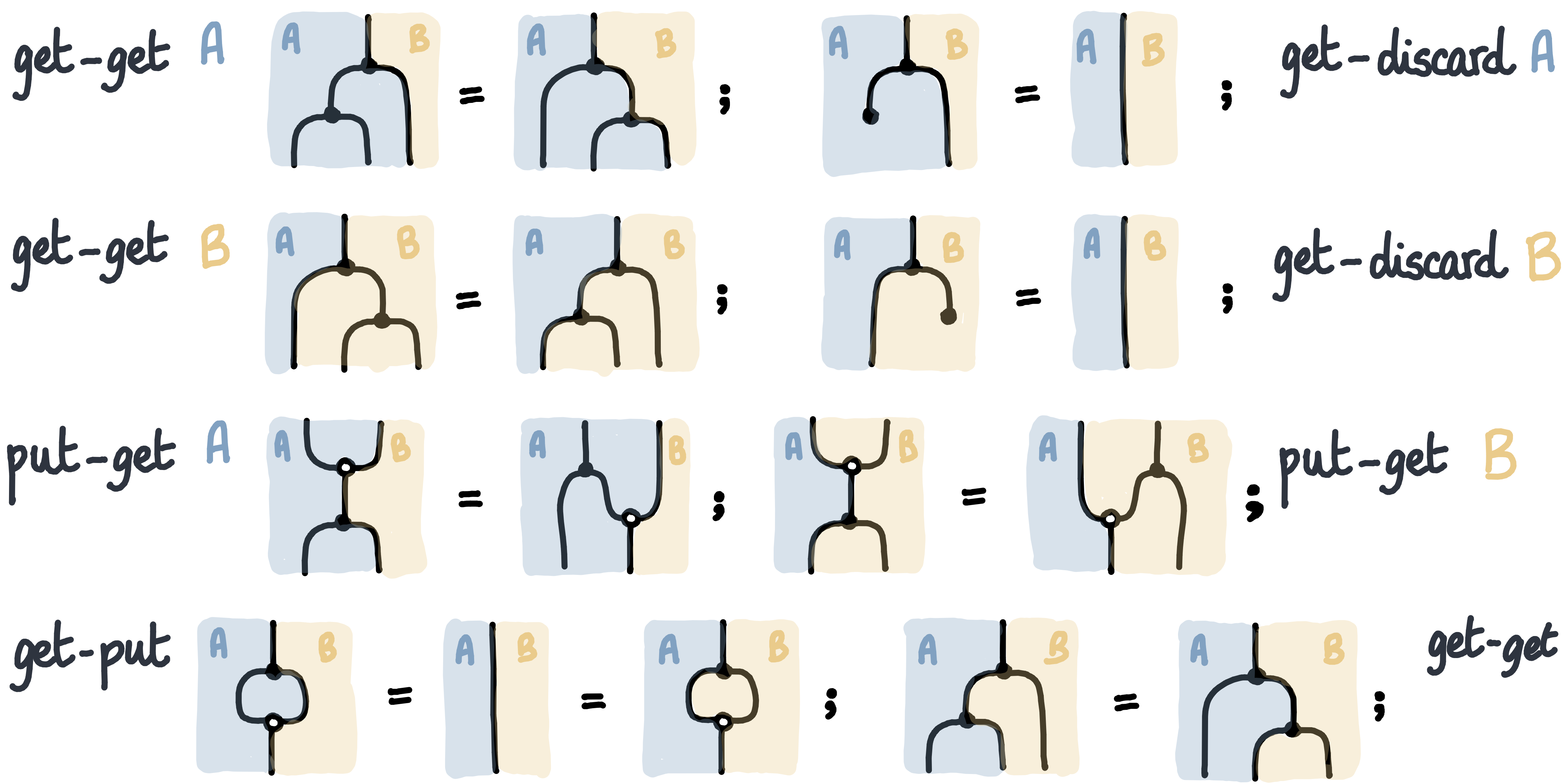


Shared State: Equations

We impose reasonable equations: e.g. overwriting.

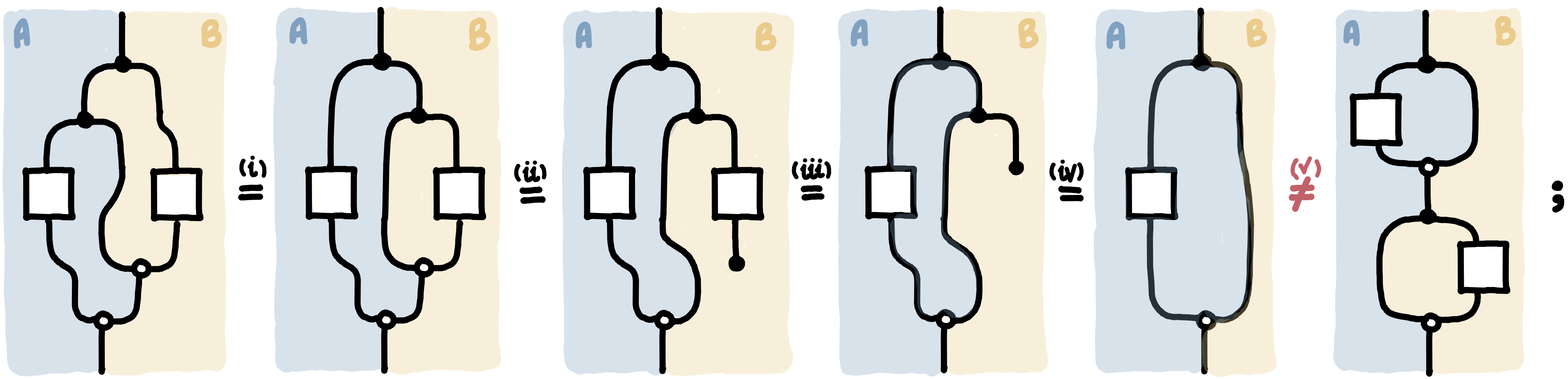


SHARED STATE: EQUATIONS



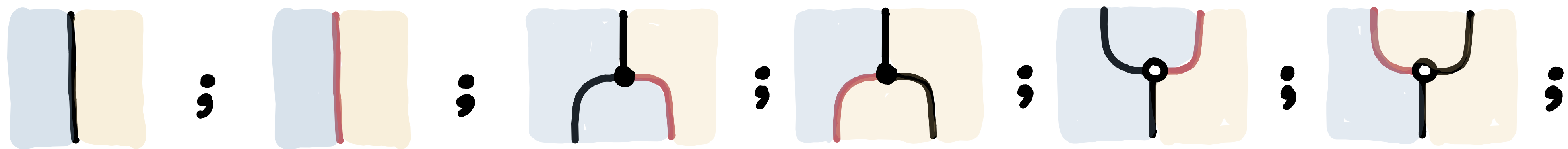
SHARED STATE: RACE CONDITION

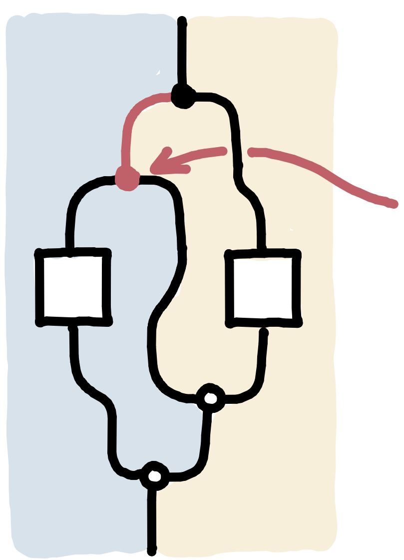
We can reproduce expected results: e.g. race condition.



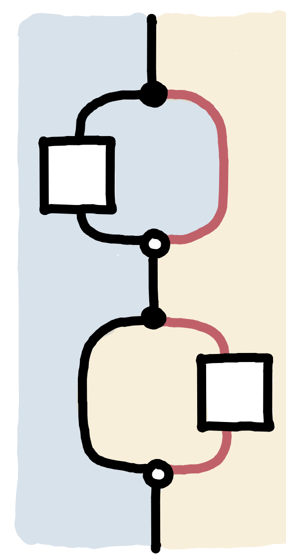
Shared State: Locks

And we can impose locks.





Race conditions are ill-typed.



Real exchanges still well-typed.

Bimodular Categories

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Bimodule	Bimodular Category

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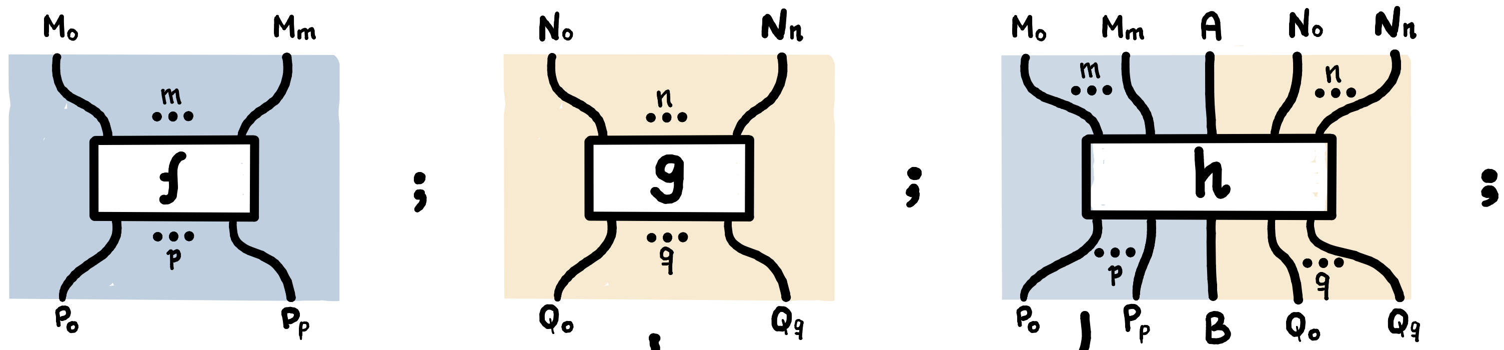
$$I \triangleright X \cong X ; \qquad X \triangleleft I \cong X ;$$

$$(M \triangleright X) \triangleleft N \cong M \triangleright (X \triangleleft N).$$

Bimodular String Diagrams

Relative to the string diagrams of a bicategory.



$f \in \mathbb{A}(M_0 \otimes \cdots \otimes M_m ; P_0 \otimes \cdots \otimes P_p)$

$h \in \mathbb{X}((M_0 \otimes \cdots \otimes M_m) \triangleright A \triangleleft (N_0 \otimes \cdots \otimes N_n))$

$g \in \mathbb{B}(N_0 \otimes \cdots \otimes N_n ; Q_0 \otimes \cdots \otimes Q_q)$

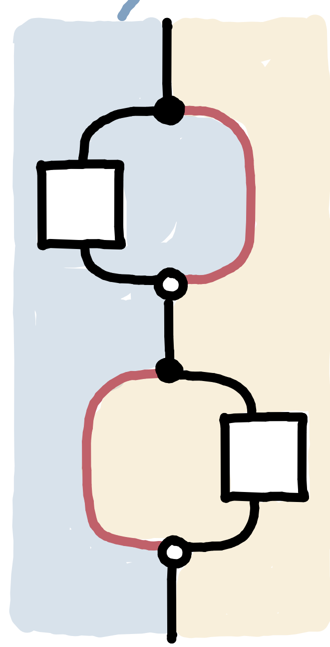
THM (BR'23). String diagrams for bimodulars are sound and complete. There is an adjunction between signatures and strict bimodulars.

2. COLLAGES

COLLAGES

We glue together string diagrams of different categories.

EXAMPLE: BIMODULARS.



String diagram in $\mathbb{A}$.

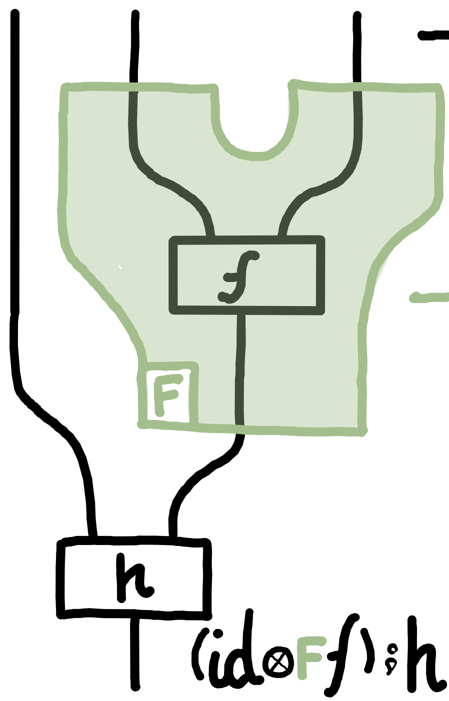
String diagram in $\mathbb{B}$.

Glued along $\mathbb{X}$, in which both can act.

COLLAGES

We glue together string diagrams of different categories.

EXAMPLE: FUNCTOR BOXES.



In some category $\mathbb{A}$

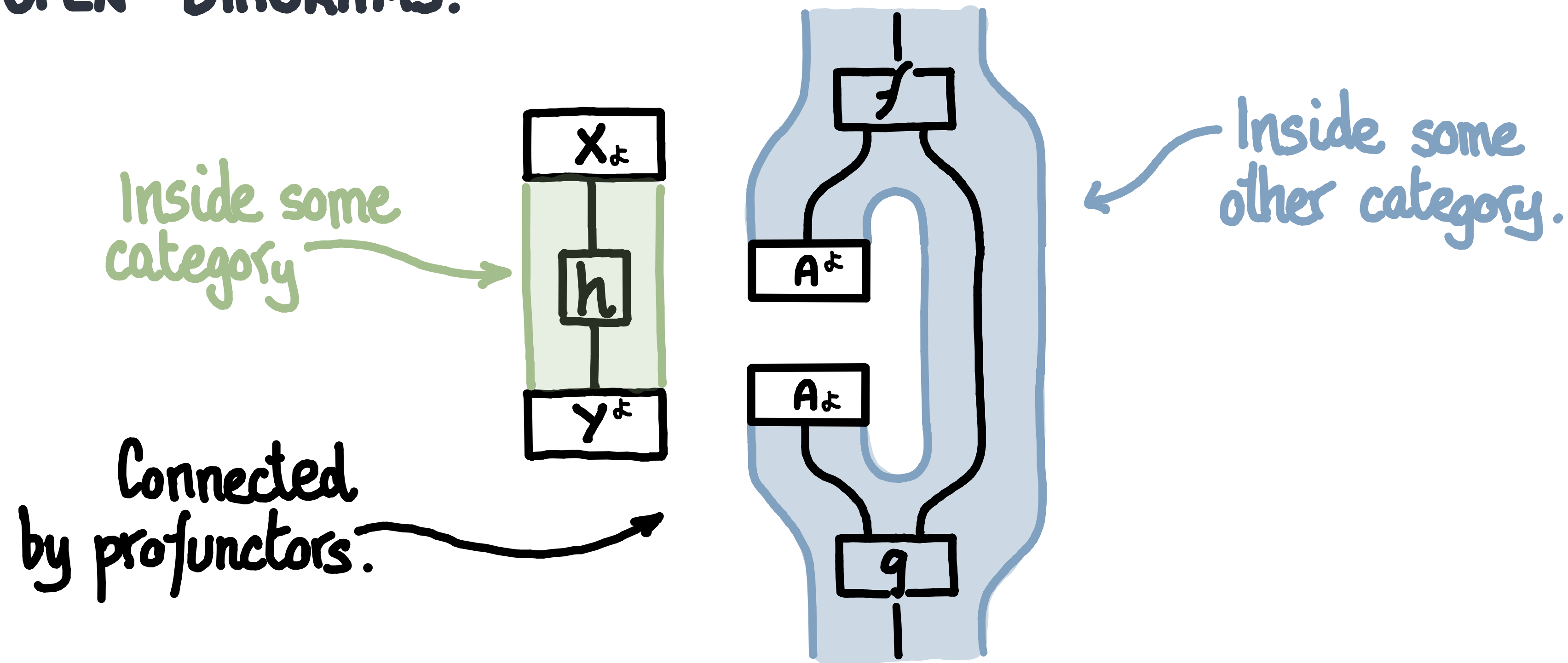
Inside some category $\mathbb{X}$

Glued along a $F : \mathbb{X} \to \mathbb{A}$ lax monoidal functor

COLLAGES

We glue together string diagrams of different categories.

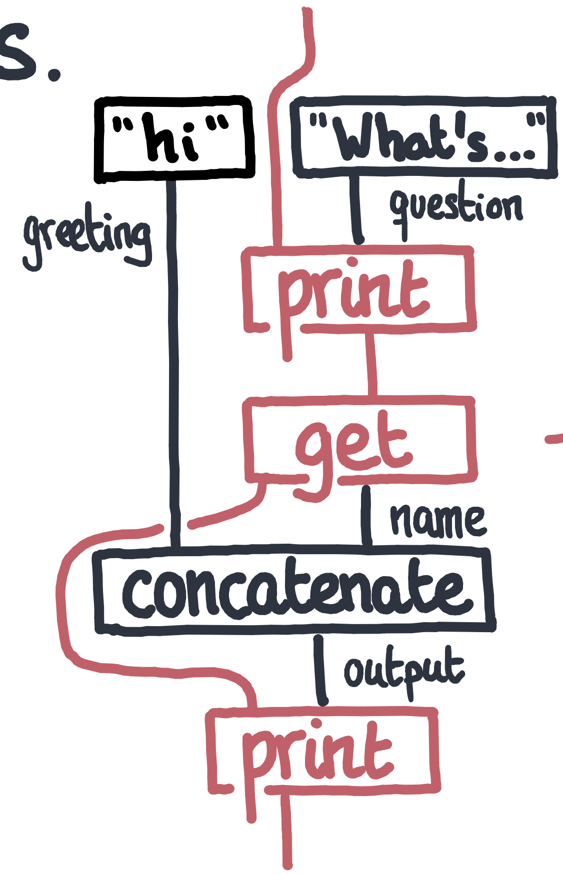
EXAMPLE: OPEN DIAGRAMS.



COLLAGES

We glue together string diagrams of different categories.

EXAMPLE: PREMONOIDALS.



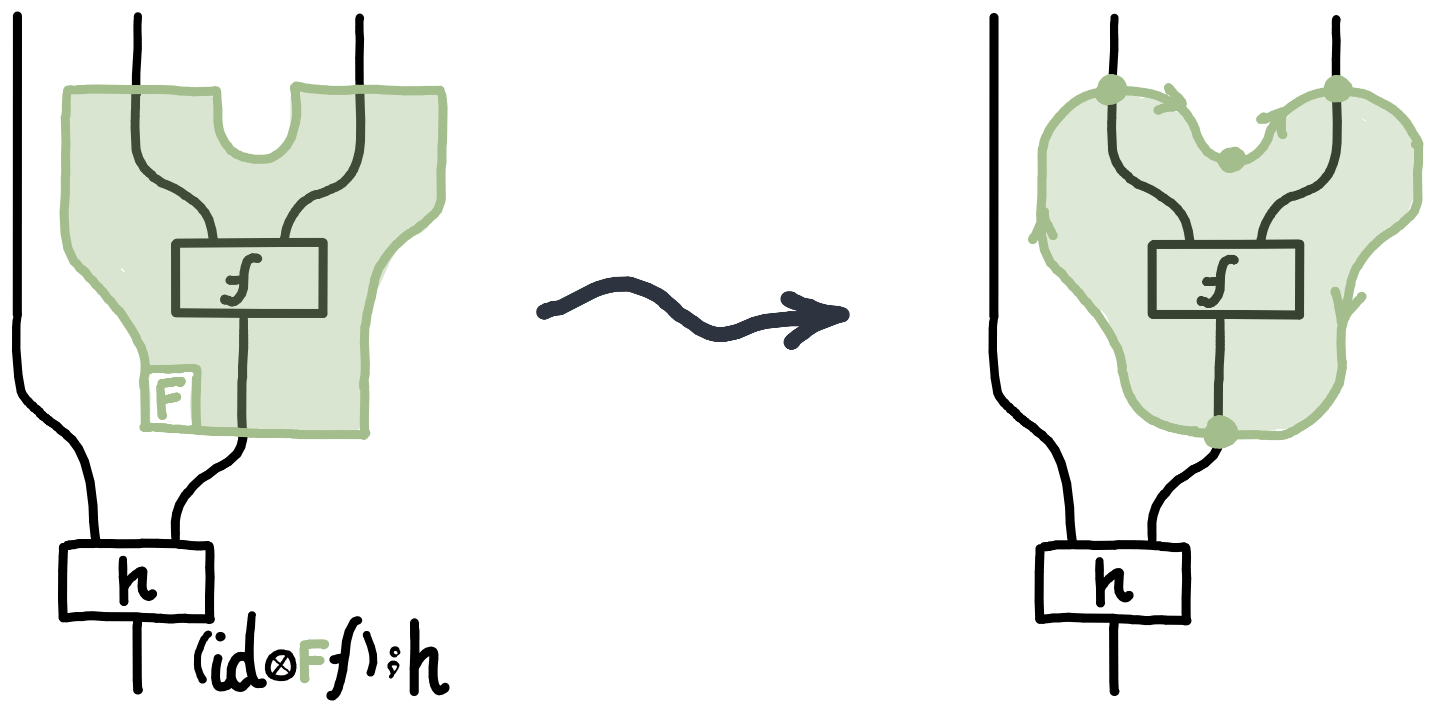
Everything inside of a monoidal category.

Except for a single wire, in the premonoidal category.

3. Everything is Strings

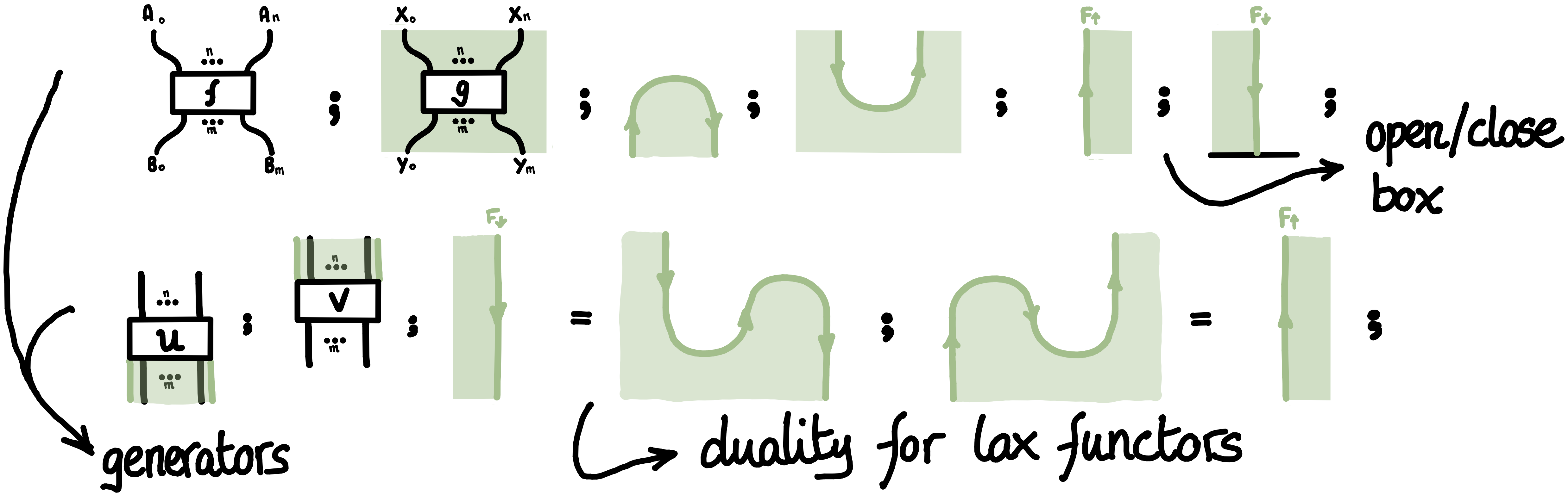
Functor Boxes Are Strings

Are functor box diagrams just diagrams in a bicategory?



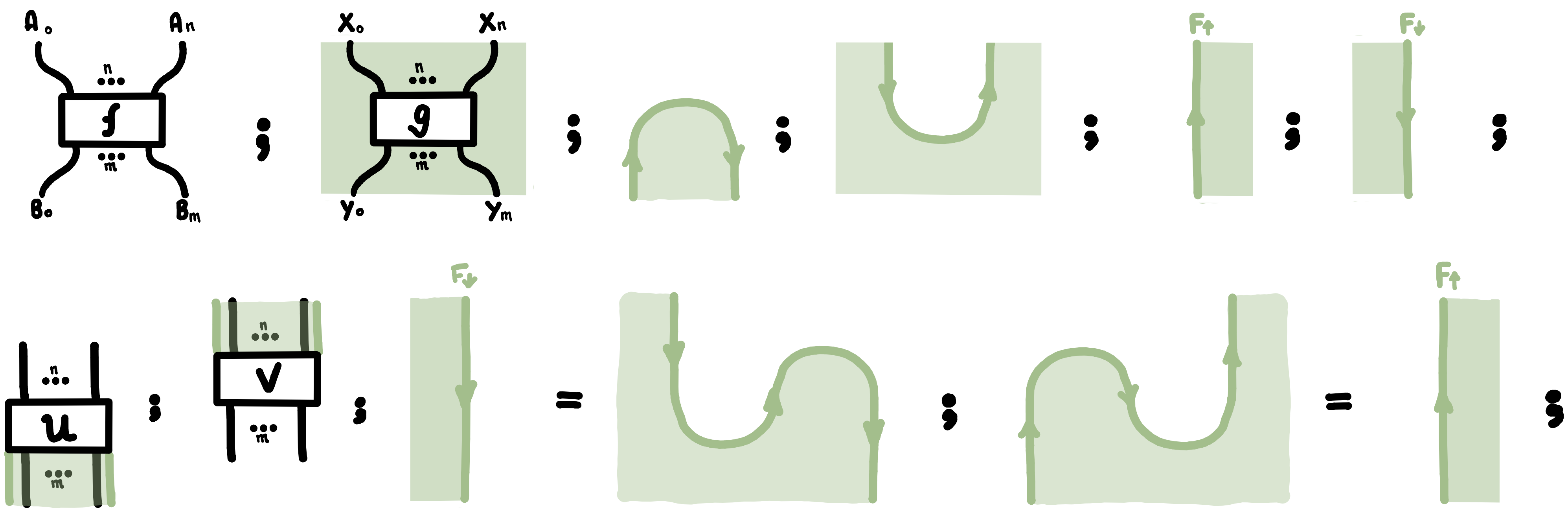
FUNCTOR BOXES ARE STRINGS

Theory of functor boxes, in bicategorical string diagrams.



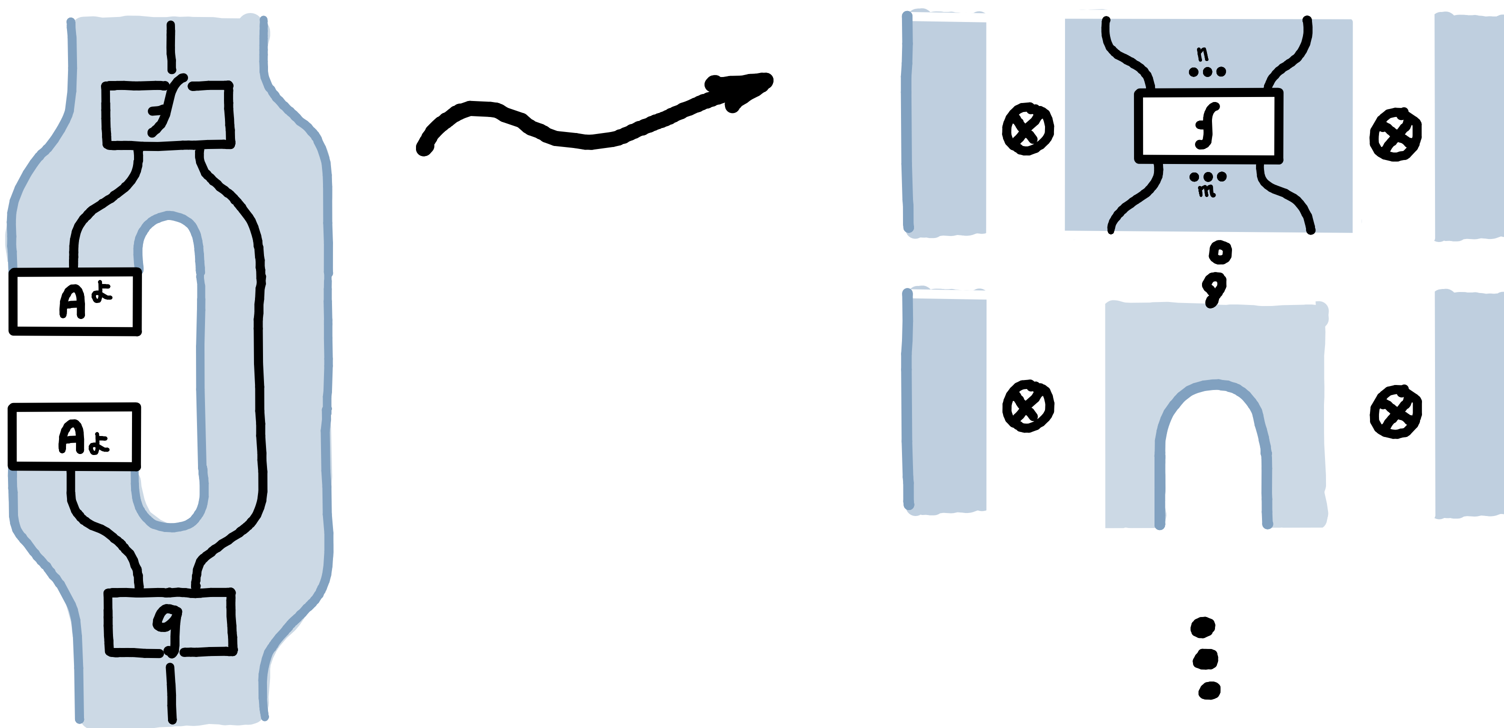
FUNCTOR BOXES ARE STRINGS

Theory of functor boxes, in bicategorical string diagrams.



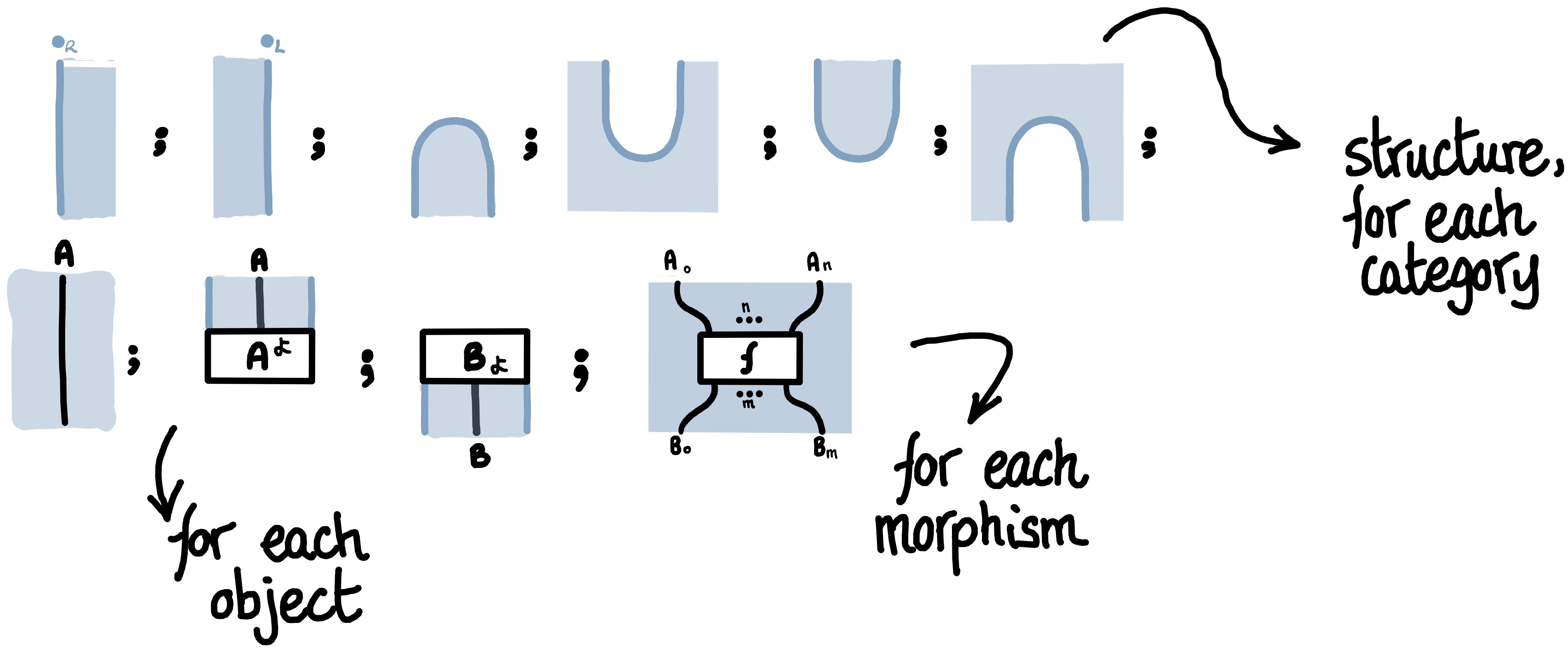
THM (BR'23). String diagrams for functor boxes are sound and complete.

Combs Are Strings



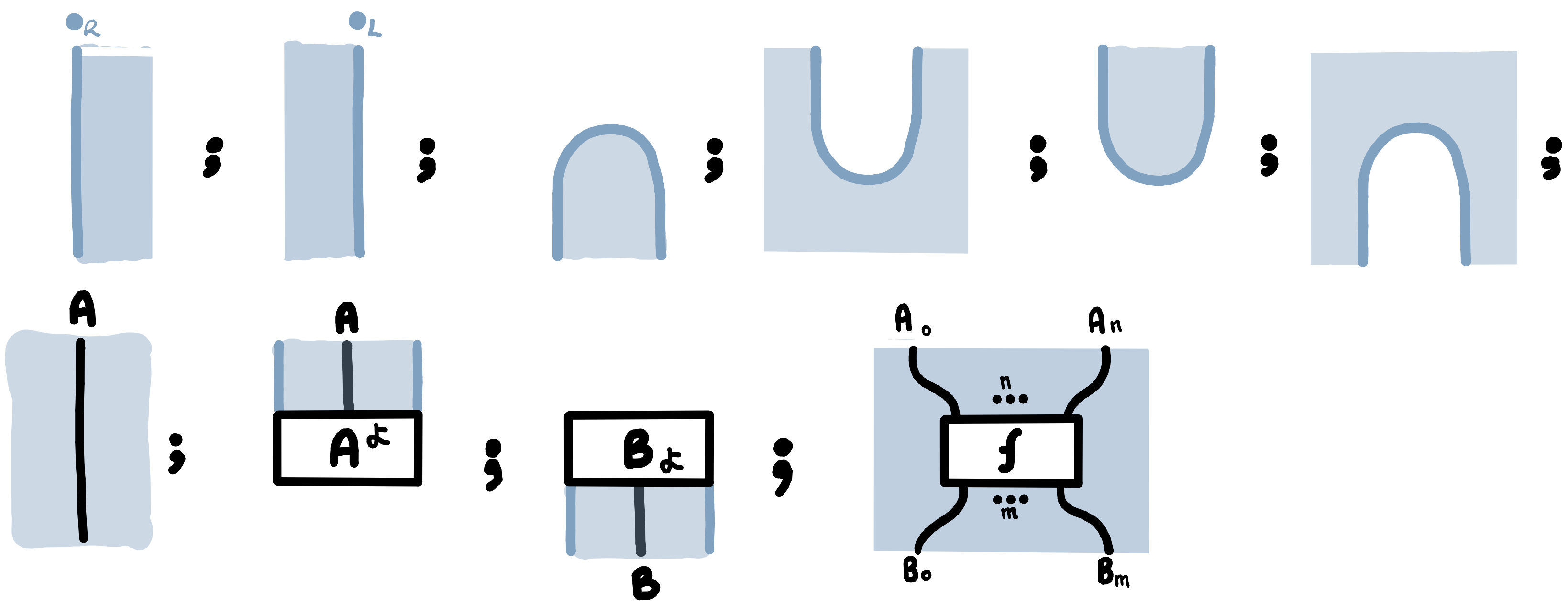
Combs Are Strings

Theory of incomplete diagrams, in terms of a 2?-category.

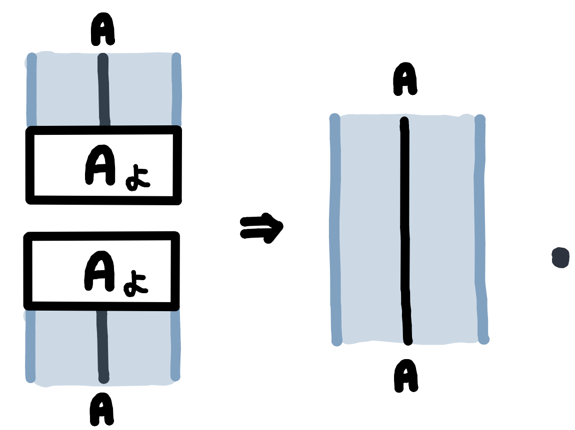


Combs Are Strings

Theory of incomplete diagrams, in terms of a 3-category.

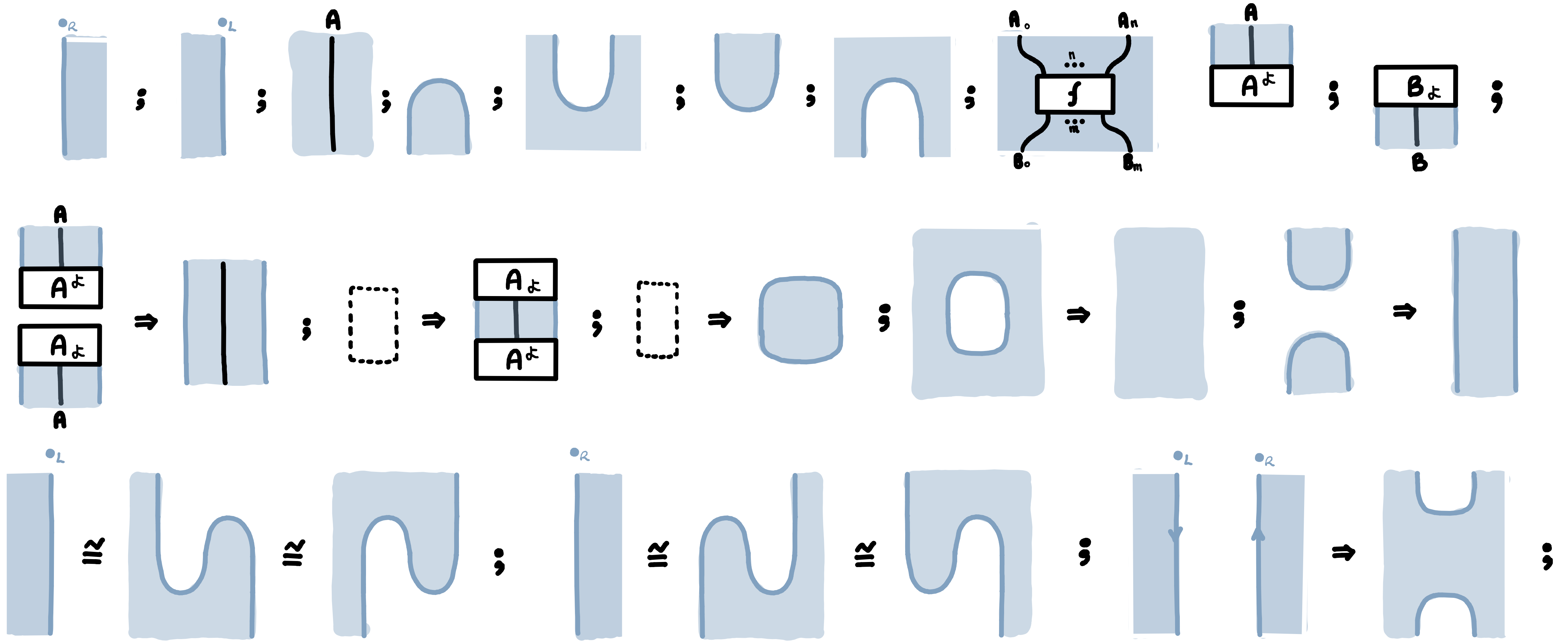


For instance,



- Reductions are also important.

Combs Are Strings



Combs Are Strings

Conjecture.
It seems plausible that we define a signature with categories, functors, profunctors, objects and morphisms and we prove soundness and completeness using 3-cat. surface/wire diagrams. These should be similar to Bartlett's Wiring Diagrams for monoidal bicats.

- What is the tricategory we are trying to model?

4. POINTED BIMODULAR PROFUNCTORS

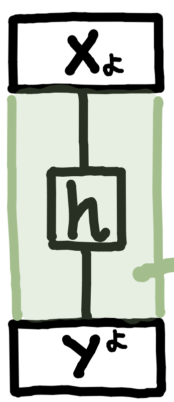
Pointed Bimodular Profunctors

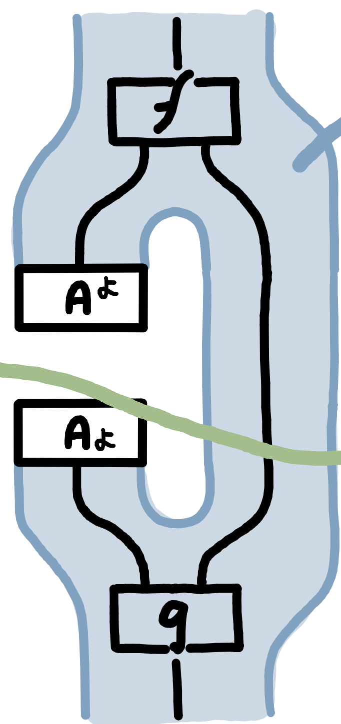
- 0-cells are monoidal categories;
- 1-cells are pointed bimodular categories, $({}_M\mathbb{X}_N, X)$, consisting on a category $\mathbb{A}$ with two monoidal actions that are compatible; and of some object $X \in \mathbb{X}$;
- 2-cells are pointed bimod. profunctors, $T_t : ({}_M\mathbb{X}_N, X_0) \nrightarrow ({}_M\mathbb{Y}_N, Y_0)$, profunctors together with a point $t \in T(X_0, Y_0)$ and compatible transformations

$$t_\ell^M : T(X;Y) \longrightarrow T(M \triangleright X; M \triangleright Y),$$
$$t_r^N : T(X;Y) \longrightarrow T(X \triangleleft N; Y \triangleleft N).$$

- 3-cells are homomorphisms of bimodular profunctors that preserve the point.

What are the Regions?

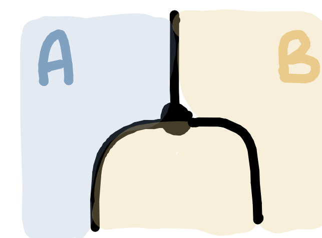


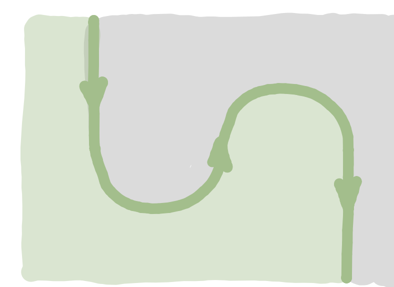


Wish:

Regions are monoidal categories.

Including the empty region.

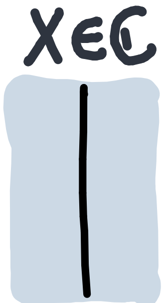




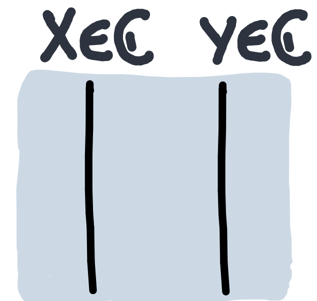
Pointed Bimodular Profunctors

- 0-cells are monoidal categories ;
- 1-cells are
- 2-cells are
- 3-cells are

WHAT ARE THE WIRES?



Objects of the monoidal category can be wires.



They tensor as in a monoidal category, $(X \in \mathbb{C}) \otimes (Y \in \mathbb{C}) \cong (X \otimes Y \in \mathbb{C})$.

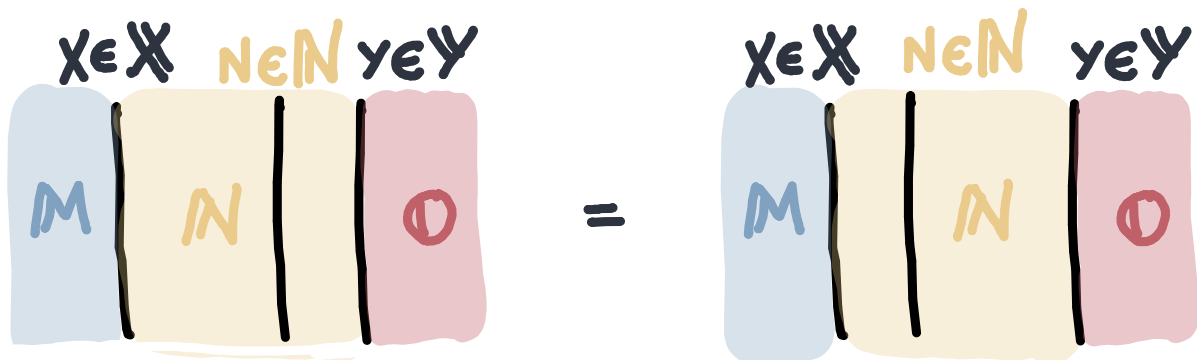
Thus, a wire between two regions must admit tensoring with wires from both categories.

Tensoring uses the bimodular structure.

$$(A \in \mathbb{A}) \otimes (X \in \mathbb{C}) \otimes (B \in \mathbb{D}) = (A \triangleright X \triangleleft B \in \mathbb{C})$$

Tensor of Bimodulars

DEF. Let ${}_{\mathbb{M}}\mathbb{X}_{\mathbb{N}}$ be a bimodular category and let ${}_{\mathbb{N}}\mathbb{Y}_{\mathbb{O}}$ be another bimodular category. Their tensor, ${}_{\mathbb{M}}\mathbb{X}_{\mathbb{N}} \otimes {}_{\mathbb{N}}\mathbb{Y}_{\mathbb{O}}$, is a category with the same objects as $\mathbb{X} \times \mathbb{Y}$ and presented by the morphisms of $\mathbb{X} \times \mathbb{Y}$ and an additional family of **equilibrators**,



$$\tau : X \otimes (N \triangleright Y) \cong (X \triangleleft N) \otimes Y,$$

satisfying coherence equations.

Pointed Bimodular Profunctors

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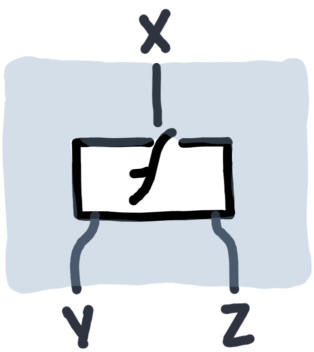
- 2-cells are
- 3-cells are

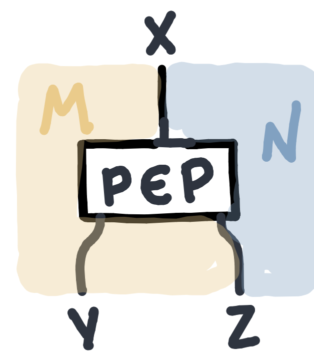
WHAT ARE THE CELLS?



$X \in \mathbb{C}$

$f \in \hom(X ; Y \otimes Z)$

$Y \otimes Z \in \mathbb{C}$



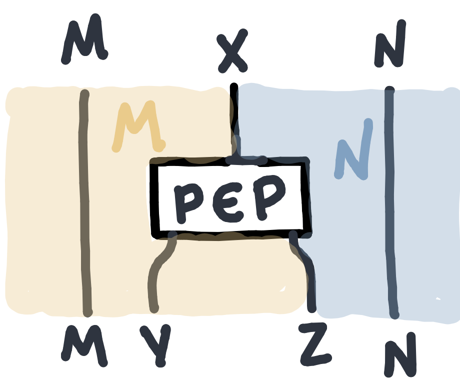
$X \in {}_{M}\mathbb{X}_{N}$

$p \in P(X ; Y \otimes Z)$

$(Y \triangleright Z) \in {}_{M}\mathbb{Y}_{N}$

Hom is a profunctor $\mathbb{C} \nrightarrow \mathbb{C}$.

P is a profunctor between bimodular categories.



Profunctors can be tensored with objects of the category, they are **Bimodular profunctors.**

Pointed Bimodular Profunctors

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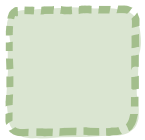
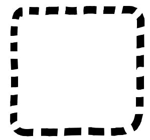
$$t_l^M : T(X;Y) \longrightarrow T(M \triangleright X; M \triangleright Y),$$
$$t_r^N : T(X;Y) \longrightarrow T(X \triangleleft N; Y \triangleleft N).$$

- 3-cells are homomorphisms of bimodular profunctors that preserve the point.

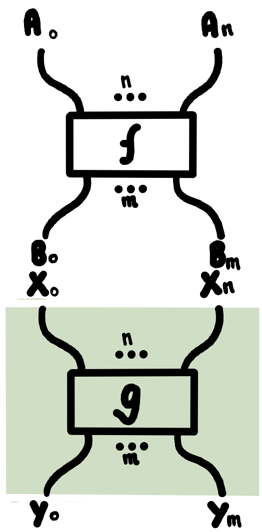
Semantics for Functor Boxes



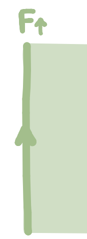
$\mathbb{X}$;



$\mathbb{A}$;

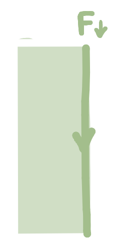


$f \in \mathbb{A}(A_0 \otimes \cdots \otimes A_n ; B_0 \otimes \cdots \otimes B_m)$;



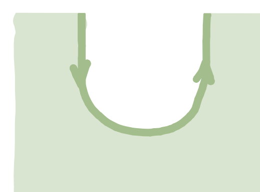
$(I ; J) \in \mathbb{A} \rtimes_F \mathbb{X}$;

$g \in \mathbb{X}(X_0 \otimes \cdots \otimes X_n ; Y_0 \otimes \cdots \otimes Y_m)$;

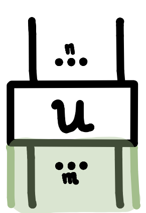


$(J ; I) \in \mathbb{X} \ltimes_F \mathbb{A}$;

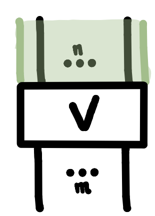
$\psi_0 \in \mathbb{A}(I ; I \otimes FJ \otimes I)$;



$(id, id) \in \mathbb{X}(J \otimes J ; J) \times \mathbb{A}(I ; I)$;

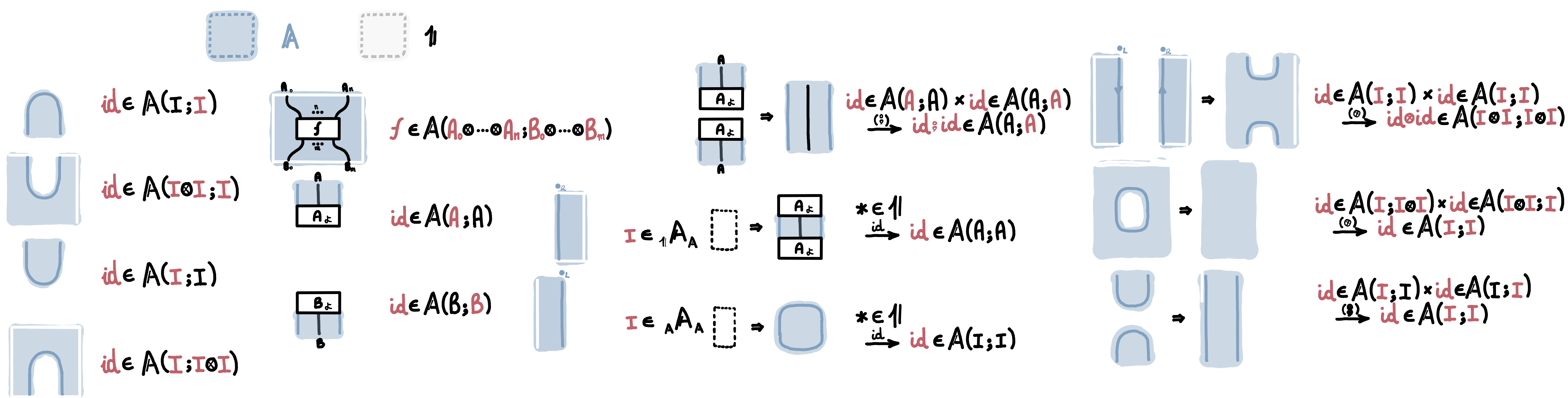


$u \in \mathbb{A}(A_0 \otimes \cdots \otimes A_n ; F(Y_0 \otimes \cdots \otimes Y_m))$;



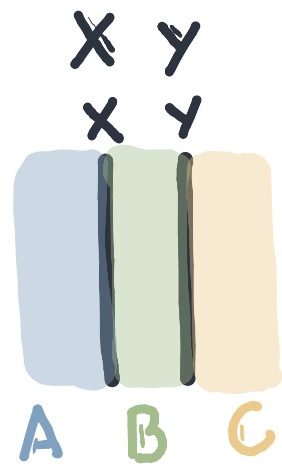
$v \in \mathbb{A}(F(X_0 \otimes \cdots \otimes X_n) ; B_0 \otimes \cdots \otimes B_m)$;

Semantics for Incomplete Diagrams



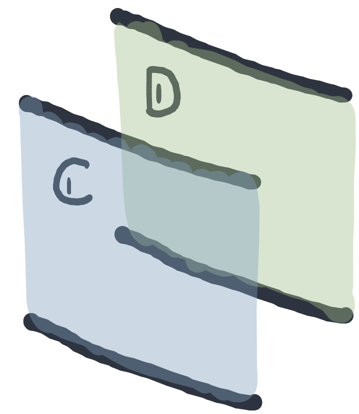
5. FURTHER

Compact Tricategory



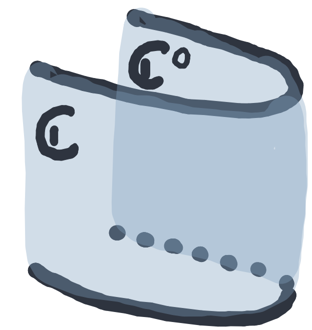
$X \otimes_{\mathbb{B}} Y, \triangleright_{\mathbb{A}}, \triangleleft_{\mathbb{C}}$

Tensor product of bimodules



$\mathbb{C} \times \mathbb{D}, \otimes_{\mathbb{C}} \times \otimes_{\mathbb{D}}, I_{\mathbb{C}} \times I_{\mathbb{D}}$

Product of monoidal categories



$\mathbb{C}^{\circ}, A \otimes^{\circ} B = B \otimes A$

Opposite Monoidal Category

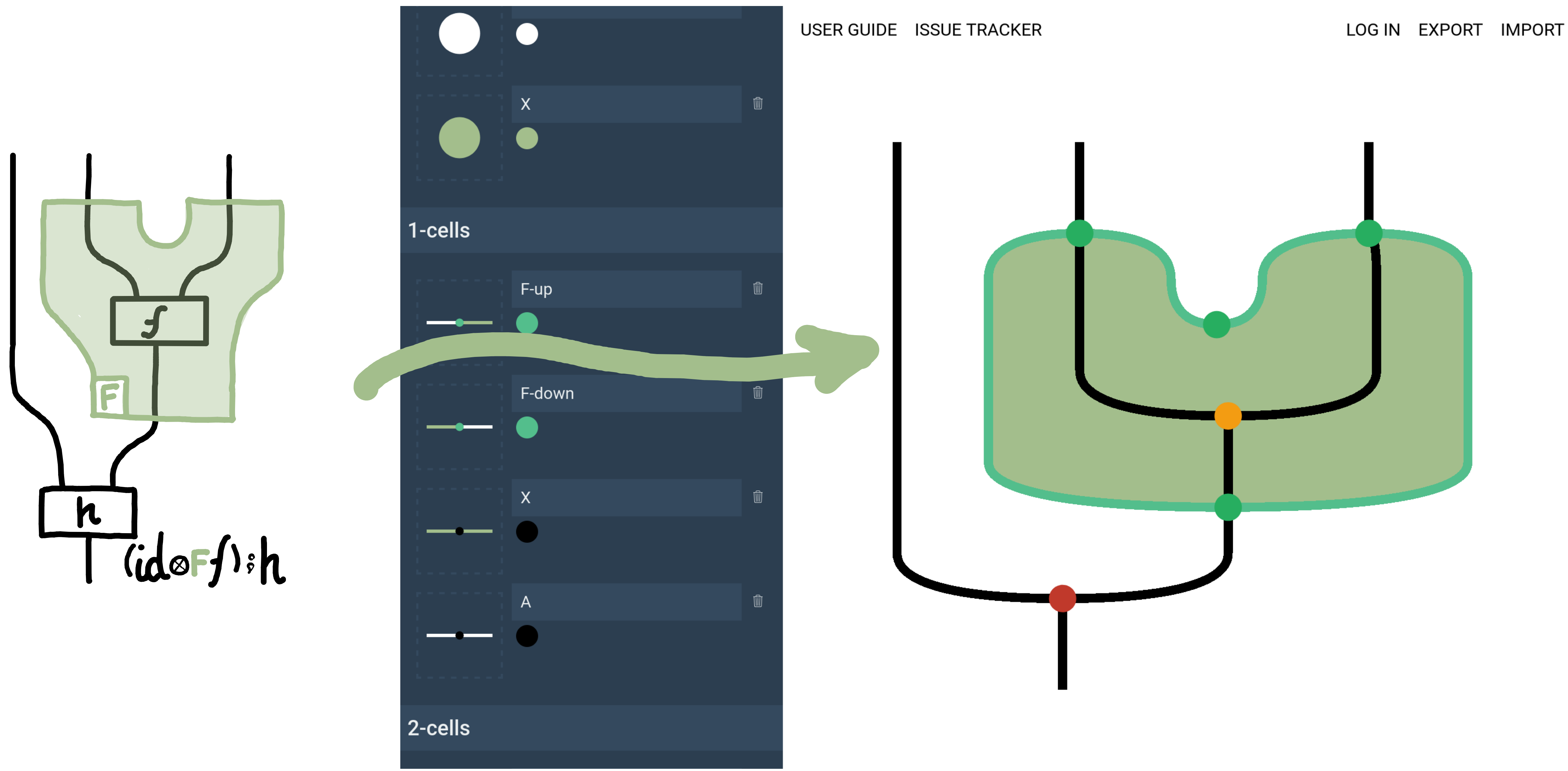
Applied Collage Theory

- Implementing bicategories feels only slightly more difficult than monoidal categories (non-symmetric).
- But these allow sound and complete calculi for functor boxes, premonoidal categories, bimodular categories, or internal diagrams.

Applied Collage Theory

Heidemann, Hu, Vicary

Homotopy.io



End

String Diagrams of String Diagrams

There has been recent interest in string diagrams of string diagrams.

- Bonchi et al. use them for bimonoidal categories;
- Zanasi et al. use them for interventions and "layered explanations";
- Vicary et al. for categorical quantum.

There are no good semantics for these. Arguably, the technology is not here yet. We would need monoidal 3-categories, with coherence conditions.

- We could impose strictness conditions.
- Instead, I want to explore this calculus from the future.

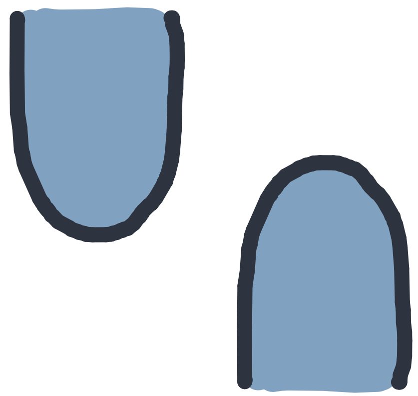
STRING DIAGRAMS OF STRING DIAGRAMS

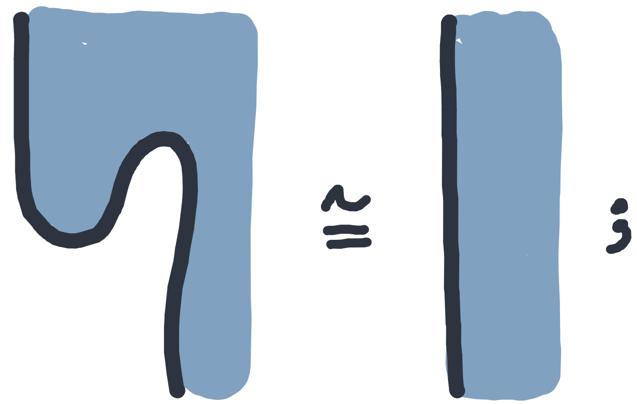
There has been recent interest in string diagrams of string diagrams.

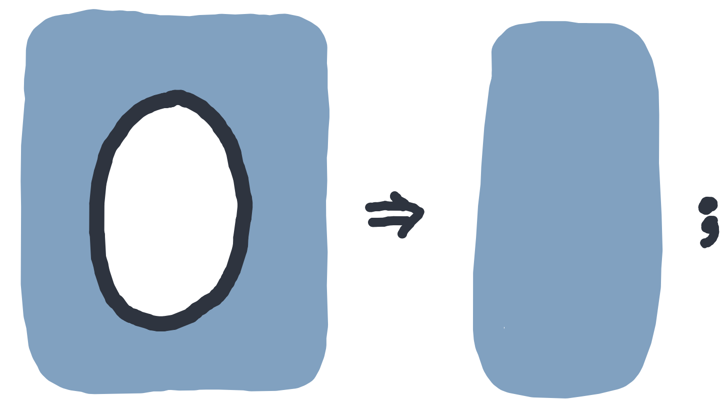
- Bonchi & Di Giorgio use them for bimonoidal categories;
- Vicary et al. for categorical quantum field theory;
- Lobski, Zanasi et al. for interventions and "layered explanations";
- M.R.: open diagrams are pointed profunctors;

- recover diagrams of optics (Riley, Hedges et al.);
- recover combs (Coecke, Fritz, Spekkens);
- recover functor boxes (Cockett & Seely and Melliès);
- recover 'negative' region (Sobocinski & Haydon, Bonchi & Di Giorgio).

Compact Tricategory







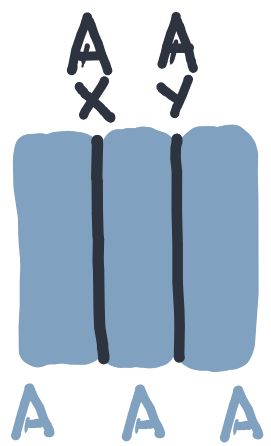
$\mathrm{id}, \hom(\cdot, I) : (\mathbb{A}, I) \nrightarrow (1, *)$

Caps and cups for internal diagrams.

2-adjunction of diagram borders

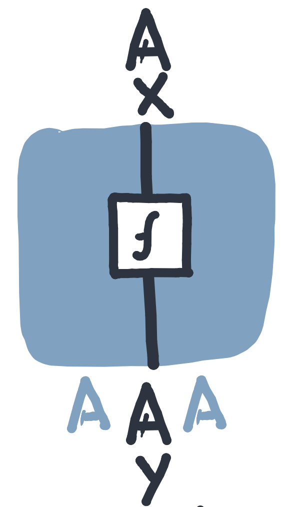
Adjunction on tensors

Compact Tricategory



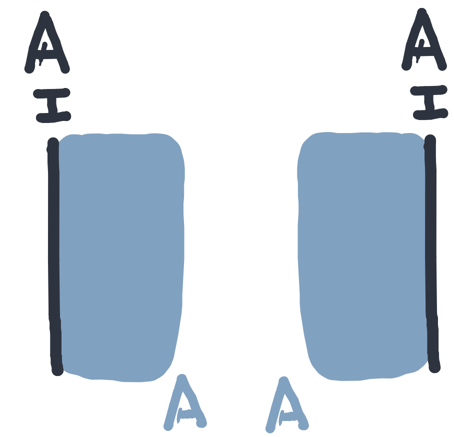
$A \otimes_A A \cong A$

Tensoring inside a monoidal category.



$f, \hom : A,X \nrightarrow A,Y$

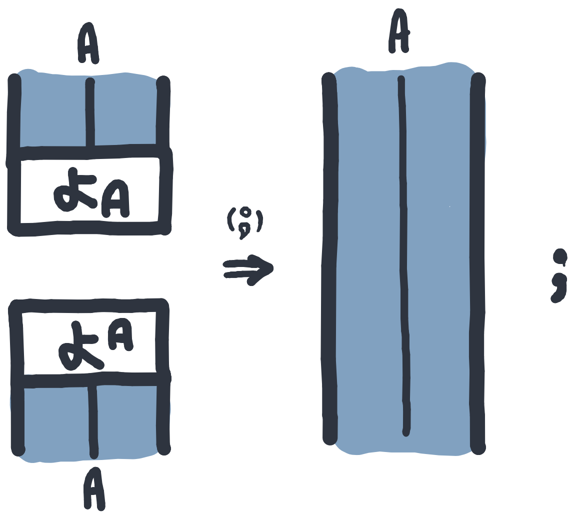
Hom is still a Tambara Module



A is a 1-bimodule

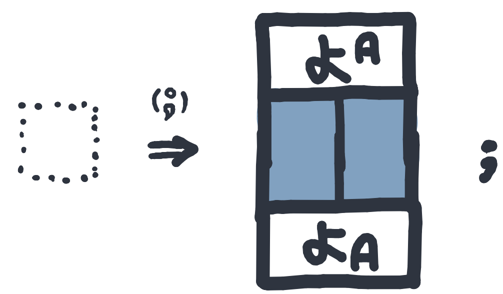
We can stop and start pieces of paper.

Compact Tricategory

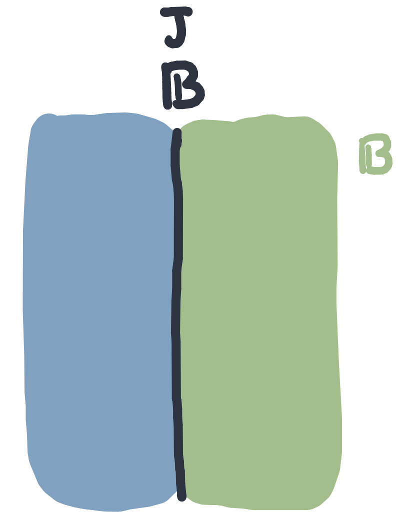


$\mathrm{id}, よ_A : \mathbb{A}, A \nrightarrow \mathbb{1}, *$

Yoneda Embeddings,
Composition



Identity function.



$(F\cdot \otimes \cdot) : \mathbb{A} \times \mathbb{B} \to \mathbb{B}$
$(\cdot \otimes \cdot) : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

Functor Boxes

Bimodular String Diagrams

THEOREM. The adjunction between bimodular signatures and bipointed 2-categories factors through bimodular categories.

$$
\begin{tikzcd}
\mathsf{bmGraph} \arrow[r, dashed, bend left] \arrow[rr, bend left=50, "i \mathbin{;} \mathsf{Str}_2"] \arrow[rr, phantom, bend left=35, "\perp"] & \mathsf{sBimod} \arrow[l, bend left, "\mathcal{U}"] \arrow[r, bend left, "\mathsf{Coll}"] \arrow[r, phantom, "\perp"] & 2\mathsf{Cat}_2 \arrow[l, bend left, "\mathsf{Chosen}"]
\end{tikzcd}
$$