Notes on "Compositional Markov"

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Part 1 : Weighted Automata

WEIGHTED AUTOMATA

DEFINITION. A weighted automaton $(Q, \gamma_Q, \delta_Q, \tau_Q) \in \mathbf{AUT}\binom{A \; X}{Y \; B}$ with parallel interfaces A, B and sequential interfaces X, Y is

- a set of states Q, with
- marked states $\gamma_Q : X \to Q$ and $\delta_Q : Y \to Q$, and
- a transition function $\tau_Q : Q \times Q \times A \times B \longrightarrow \mathbb{R}^+$.

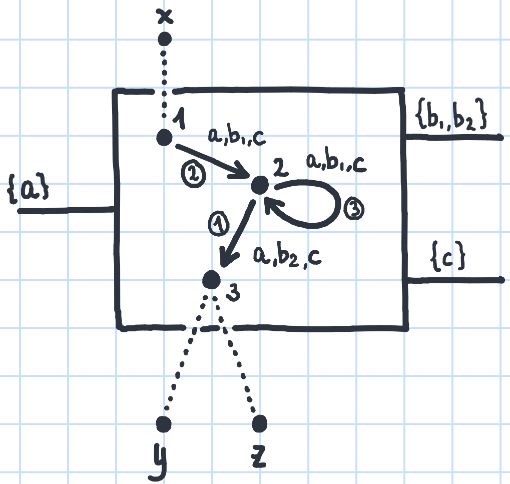
The value of $\tau_Q(q_1, q_2, a, b)$ is

the weight of a transition from q_1 to q_2 producing signals a and b.

The signals A, B are used to "sync" automata, the X, Y marked states are used to "glue" automata.

WEIGHTED AUTOMATA

Drawing weighted automata



- with states $Q=\{1,2,3\}$,
- marked states $\{1\}$ and $\{3\}$ by $X=\{x\}$ and $Y=\{y,z\}$,
- signals $A=\{a\}$, $B=\{b_1,b_2\}$ and $C=\{c\}$,
- transitions

$$\begin{array}{c|c} \text{from}\backslash\text{to} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c|c} \text{from}\backslash\text{to} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

with a,b_1,c with a,b_2,c

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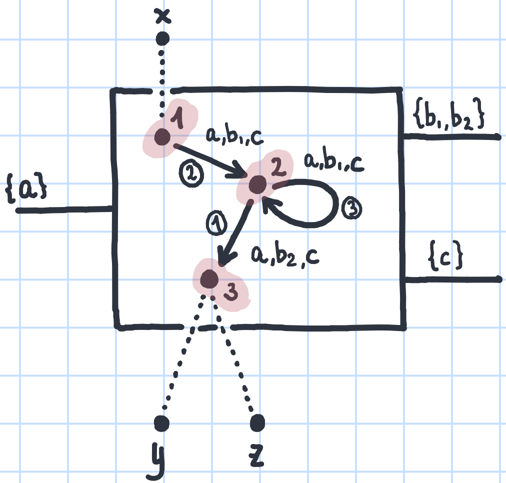

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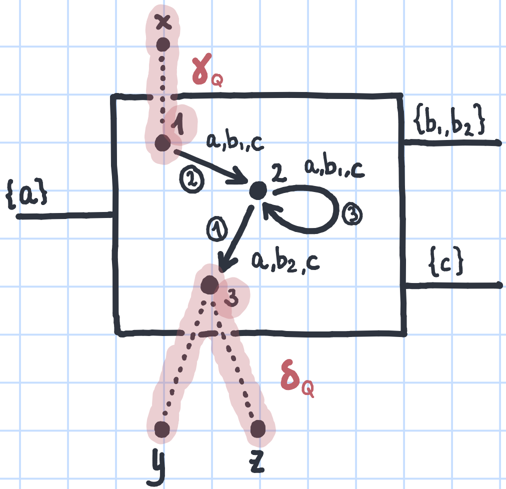

- with states $Q = \{1,2,3\}$,
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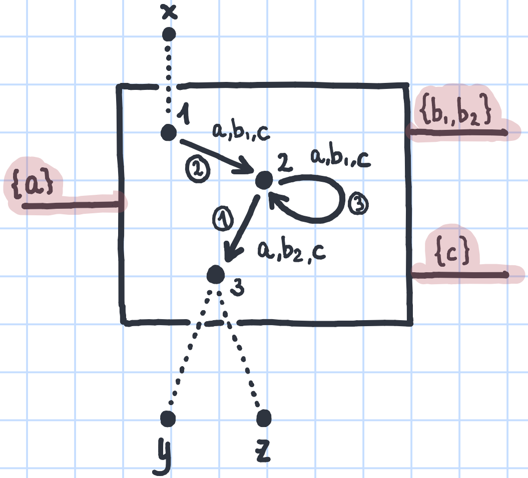
with a, b_2, c

WEIGHTED AUTOMATA

Drawing weighted automata



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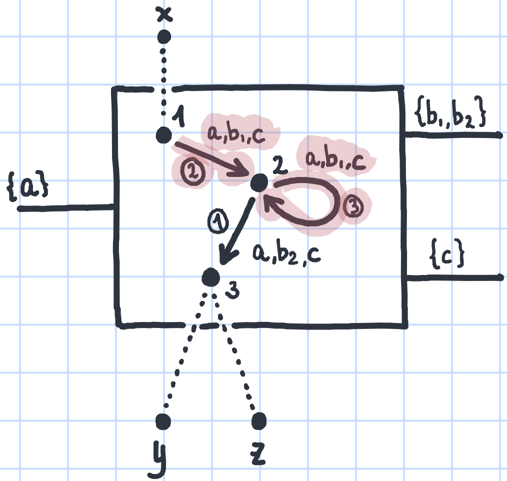

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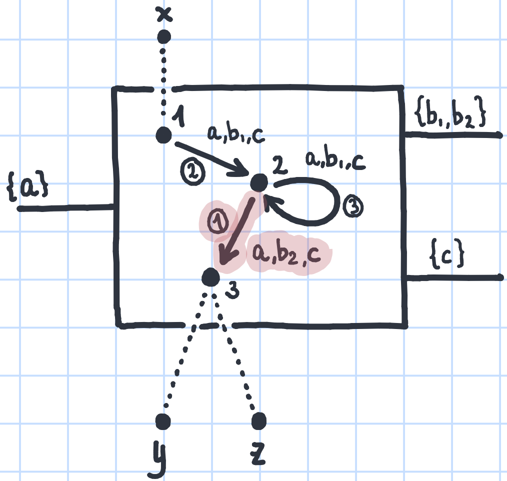
with a, b_1, c with a, b_2, c

$\tau(a, b_1, c, \cdot, \cdot)$

WEIGHTED AUTOMATA

Drawing weighted automata



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with a,b_1,c with a,b_2,c

$\tau(a,b_2,c,\cdot,\cdot)$

Part 2 : Composing Automata

In Parallel

Product

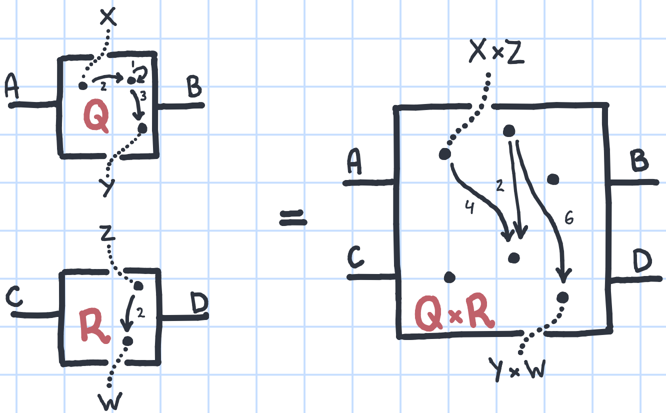
The product of

$Q = (S_Q, \gamma_Q, \delta_Q, \tau_Q) : \mathrm{Aut}(A \overset{X}{\underset{Y}{\to}} B)$ and $R = (S_R, \gamma_R, \delta_R, \tau_R) : \mathrm{Aut}(C \overset{Z}{\underset{W}{\to}} D)$

is the automaton

$Q \times R = (S_Q \times S_R,\ \gamma_Q \times \gamma_R,\ \delta_Q \times \delta_R,\ \tau_Q \cdot \tau_R) : \mathrm{Aut}(A \times C \overset{X \times Z}{\underset{Y \times W}{\to}} B \times D)$,

$\tau_Q \cdot \tau_R (a, b, c, d, q_1, q_2, r_1, r_2) = \tau_Q(a, b, q_1, q_2) \cdot \tau_R(c, d, r_1, r_2)$.



PARALLEL

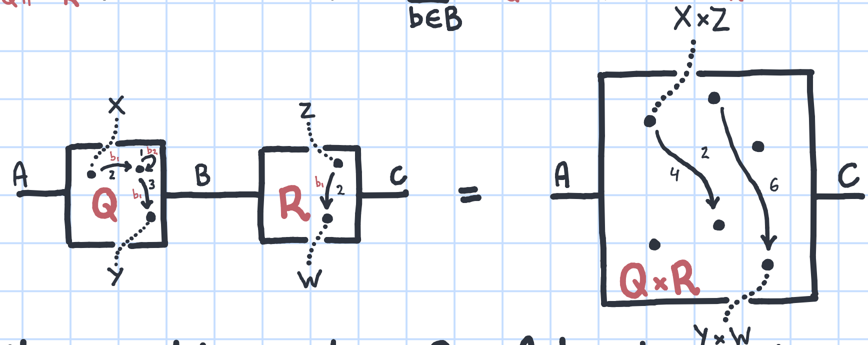
The parallel communicating composition of

$Q = (S_Q, \gamma_Q, \delta_Q, \tau_Q) : \mathrm{AUT}\left(A \overset{X}{\underset{Y}{\rightleftarrows}} B\right)$ and $R = (S_R, \gamma_R, \delta_R, \tau_R) : \mathrm{AUT}\left(B \overset{Z}{\underset{W}{\rightleftarrows}} C\right)$

is the automaton

$$Q \| R = (S_Q \times S_R, \gamma_Q \times \gamma_R, \delta_Q \times \delta_R, \tau_Q \| \tau_R) : \mathrm{AUT}\left(A \overset{X \times Z}{\underset{Y \times W}{\rightleftarrows}} C\right),$$

$$\tau_Q \| \tau_R (a, c, q_1, q_2, r_1, r_2) = \sum_{b \in B} \tau_Q(a, b, q_1, q_2) \cdot \tau_R(b, c, r_1, r_2).$$

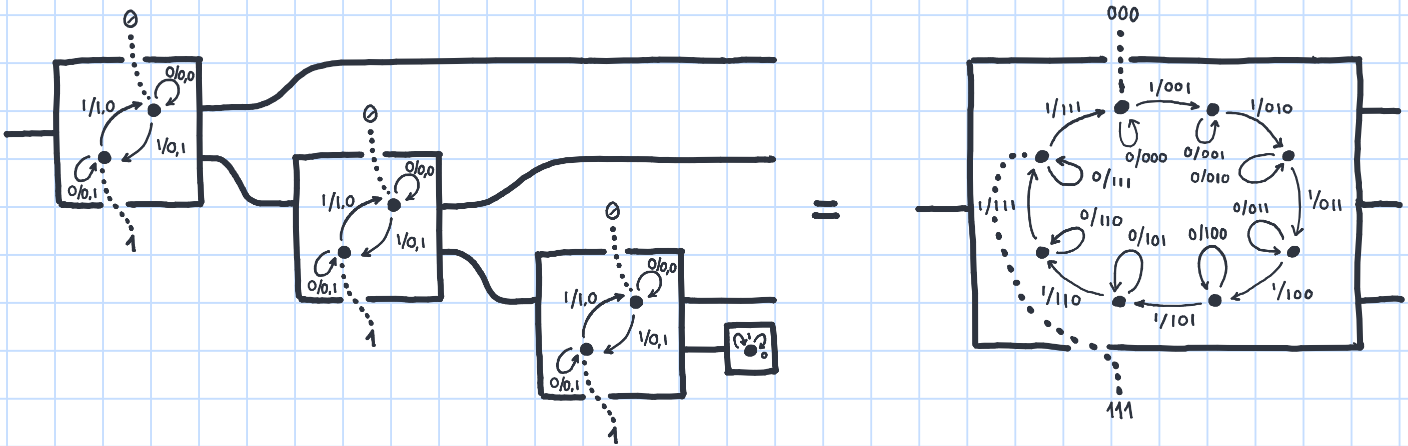


Transitions need to coincide on B. Automata synchronization.

Parallel Monoidal Category

The category ParAut has sets as objects and automata $\mathrm{Aut}(A \overset{X}{\underset{Y}{\diamond}} B)$ for some X, Y as morphisms $A \to B$. It uses parallel-communicating composition, and the product makes it a symmetric monoidal category.

Example: 3-bit binary counter.

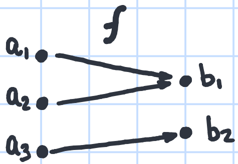


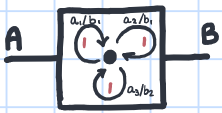
Creating Automata

Any function $f : A \to B$ can be lifted to automata $f_* : \mathrm{Aut}(A \overset{\circ}{\underset{\circ}{\ }} B)$ and $f^* : \mathrm{Aut}(B \overset{\circ}{\underset{\circ}{\ }} A)$

$$f_* = (\mathbb{1}, \phi, \phi, \tau_f), \quad \text{where} \quad \tau_f(a, f(a), *, *) = 1, \text{ and } 0 \text{ otherwise}$$
$$f^* = (\mathbb{1}, \phi, \phi, \tau^f), \quad \text{where} \quad \tau^f(f(a), a, *, *) = 1, \text{ and } 0 \text{ otherwise}$$

with a single state, no marked states and f-labelled transitions.





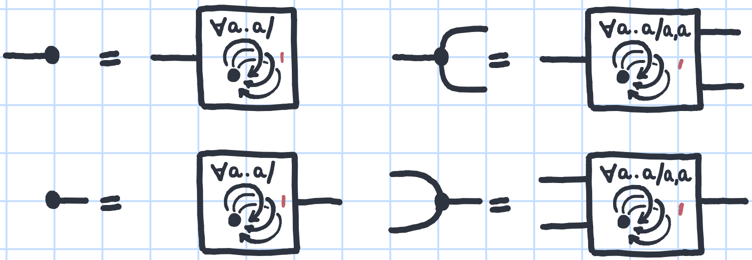
These are stateless signal transformers, $a \mapsto f(a)$.

Proposition. These liftings determine strong monoidal functors

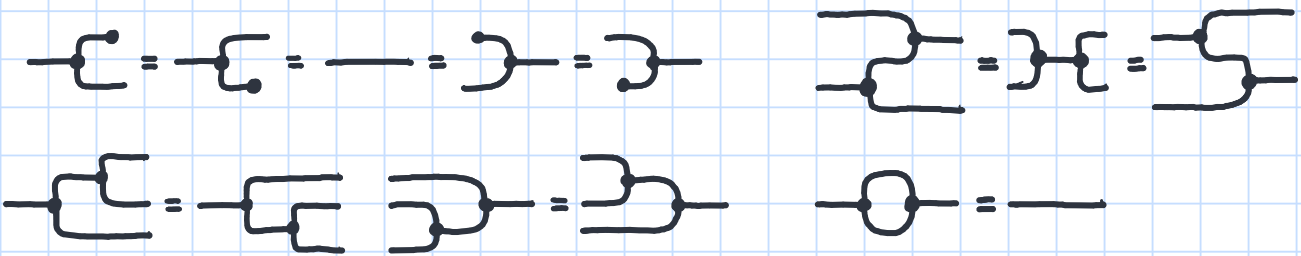
$$(\mathrm{Set}, \times) \longrightarrow (\mathrm{ParAut}, \times), \qquad (\mathrm{Set}^{op}, \times) \longrightarrow (\mathrm{ParAut}, \times).$$

PARALLEL MONOIDAL CATEGORY

We use dots for copying/discarding signals.



These form a special Frobenius monoid: "only connectivity matters".



Part 3 : Composing Automata

Sequentially

SUM

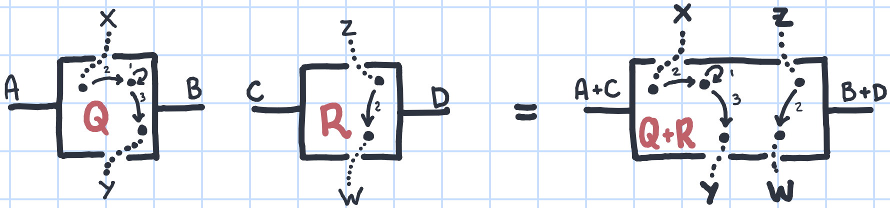
The sequential composition of

$Q=(S_Q, \gamma_Q, \delta_Q, \tau_Q) : \mathrm{AUT}\left(A \overset{X}{\underset{Y}{\circ}} B\right)$ and $R=(S_R, \gamma_R, \delta_R, \tau_R) : \mathrm{AUT}\left(C \overset{Z}{\underset{W}{\circ}} D\right)$

is the automaton

$$Q+R = (S_Q+S_R,\ \gamma_Q+\gamma_R,\ \delta_Q+\delta_R,\ \tau_Q \vee \tau_R) : \mathrm{AUT}\left(A+C \overset{X+Z}{\underset{Y+W}{\circ}} B+D\right)$$

$$\tau_Q \vee \tau_R(a_A, b_B, q_{1Q}, q_{2Q}) = \tau_Q(a,b,q_1,q_2)$$
$$\tau_Q \vee \tau_R(c_C, d_D, r_{1R}, r_{2R}) = \tau_R(c,d,r_1,r_2)$$
$$\tau_Q \vee \tau_R(\bullet, \bullet, \bullet, \bullet) = 0, \text{ otherwise.}$$



SEQUENTIAL

The sequential communicating composition of

$$Q = (S_Q, \gamma_Q, \delta_Q, \tau_Q) : \mathrm{AUT}\left(A \,{}^{X}_{Y}\, B\right) \quad \text{and} \quad R = (S_R, \gamma_R, \delta_R, \tau_R) : \mathrm{AUT}\left(C \,{}^{Y}_{Z}\, D\right)$$

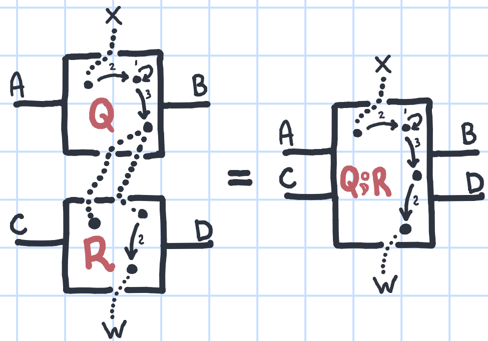
is the automaton

$$Q \mathbin{⨾} R = (S_Q +_Y S_R, \gamma_Q, \delta_R, \tau_Q \circ \tau_R) : \mathrm{AUT}\left(A + C \,{}^{X}_{Z}\, B + D\right),$$

$$\tau_Q \mathbin{⨾} \tau_R (a_A, b_B, q_{1Q}, q_{2Q}) = \sum_{\delta_Q(s_1) = q_1} \sum_{\delta_Q(s_2) = q_2} \tau_Q(a_A, b_B, s_{1Q}, s_{2Q}),$$

$$\tau_Q \mathbin{⨾} \tau_R (c_C, d_D, r_{1R}, r_{2R}) = \sum_{\gamma_R(t_1) = r_1} \sum_{\gamma_R(t_2) = r_2} \tau_R(c_C, d_D, t_{1R}, t_{2R}).$$

$$\tau_Q \mathbin{⨾} \tau_R (\bullet, \bullet, \bullet, \bullet) = 0, \text{ otherwise.}$$



SEQUENTIAL MONOIDAL CATEGORY

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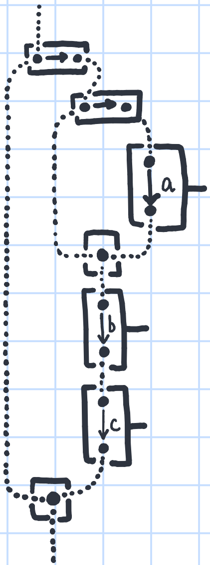

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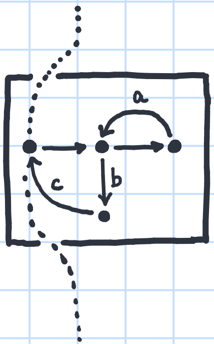


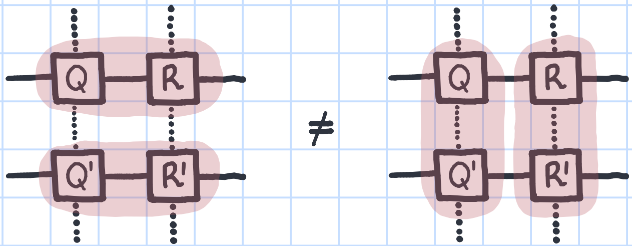
EXAMPLE: Regex $(a^* bc)^*$
Some sequences of a, always ended by (bc).

Extra

INTERCATEGORIES

- A double category of automata?

  Parallel/sequential composition do not satisfy the interchange law.



Grandis & Paré use these as an example of an intercategory.

References

Albasini, Sabadini, Walters. The Compositional Construction of Markov Processes.