GRADED COALGEBRAS OF MONADS

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Motivation

- Recent renewed interest in categorical continuous dynamical systems.
- Coalgebra has the tools for the uniform study of dynamical systems.
- And many tools for continuous analysis.
- But the perception is that coalgebras, $X \to FX$, are "discrete".

Rutten, 2005. Escardó, Pavlovic, 1998. Silva, Kozen, 2014.

Coalgebras of a Comonad

Beyond coalgebras for an endofunctor, a comonad (R, ε, δ) imposes extra equations.

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & RX \arrow[d, "R\alpha"] \\
RX \arrow[r, "\delta"'] & R(RX)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, no head] \arrow[dr, phantom, "=", shift left=2] & RX \arrow[d, "\varepsilon"] \\
 & X
\end{tikzcd}
$$

e.g. Applegate, Tierney, 1970. Jacobs, Rutten, 1997.

COALGEBRAS OF A MONAD

Can we do the same with a monad?

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & TX \arrow[d, "T\alpha"] \\ TX & T(TX) \arrow[l, "m"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\eta"'] \arrow[dr, phantom, "="] & TX \\ TX \arrow[ur, dash, "id"'] & \end{tikzcd}$$

The first is a bit restrictive, but the second makes everything collapse.

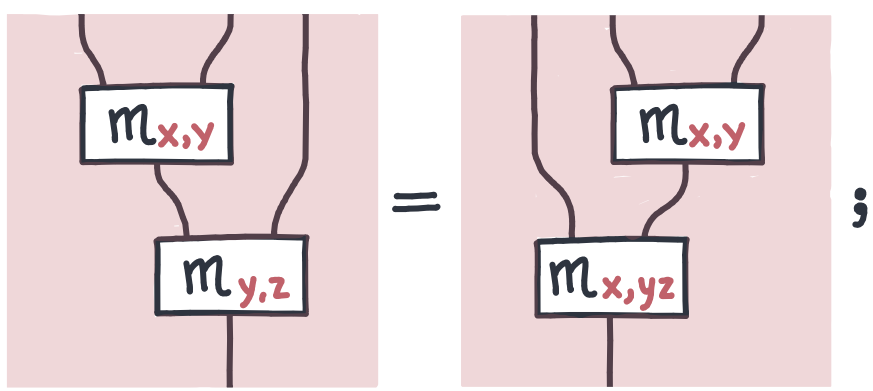
OUTLINE.

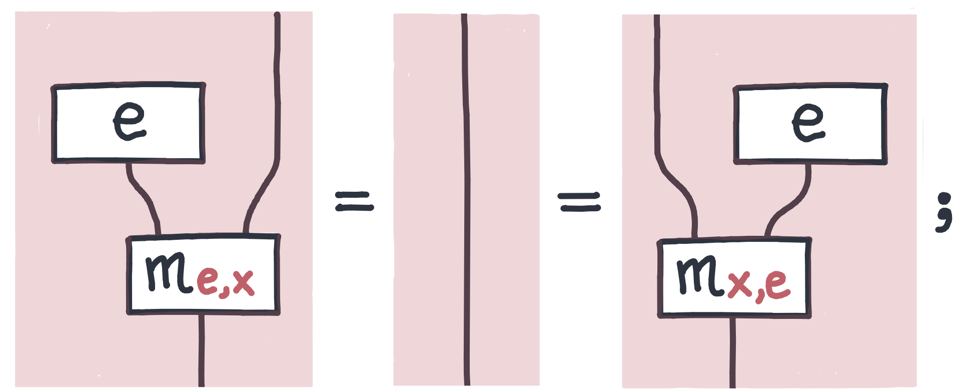
1. Graded coalgebras of a monad
2. Recovering usual examples.
3. Examples.

Graded Monads

Family of endofunctors, $T_x : \mathbb{C} \to \mathbb{C}$, graded by a monoid, with multiplication and unit, $m_{x,y} : T_x(T_y A) \to T_{x \cdot y} A$ and $\varepsilon : A \to T_e A$; following axioms.

$$\begin{tikzcd}
T_x(T_y(T_z A)) \arrow[r] \arrow[d] \arrow[dr, phantom, "//"] & T_{x\cdot y}(T_z A) \arrow[d] \\
T_x(T_{y\cdot z} A) \arrow[r] & T_{x\cdot y\cdot z} A
\end{tikzcd}
\qquad
\begin{tikzcd}
T_x A \arrow[r] \arrow[d] \arrow[dr, no head] & T_x(T_e A) \arrow[d] \\
T_e(T_x A) \arrow[r] & T_x A
\end{tikzcd}$$

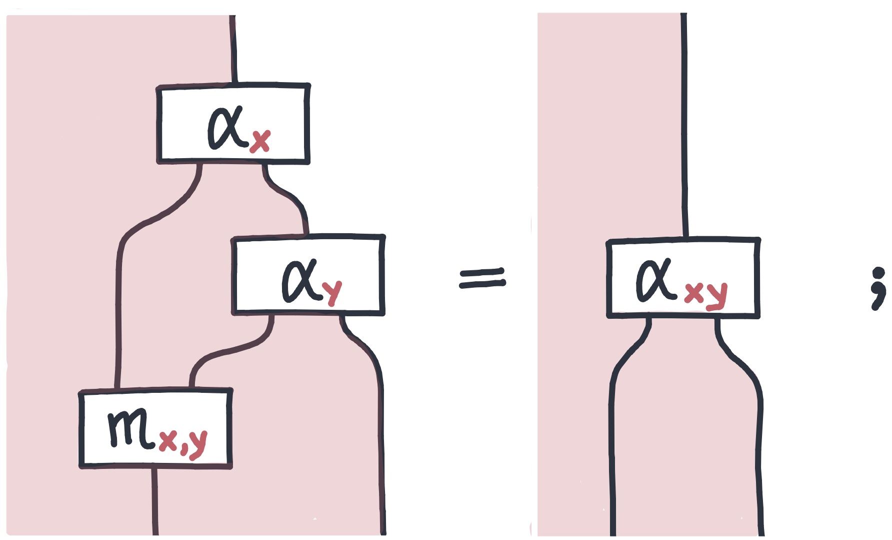


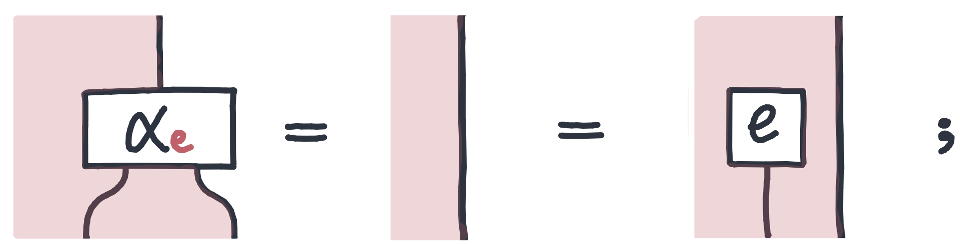


Smirnov, 2008. Fujii, Katsumata, Melliès, 2016. Milius, Pattinson, Schröder, 2020.

Graded Coalgebras of a Graded Monad

$$
\begin{tikzcd}
A \arrow[r, "\alpha_x"] \arrow[d, "\alpha_{xy}"'] \arrow[dr, phantom, "="] & T_x A \arrow[d, "T\alpha_y"] \\
T_{xy} A & T_x(T_y A) \arrow[l, "m_{x,y}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\eta"'] \arrow[dr, phantom, "="] & T_e X \arrow[dl, dash, "id"] \\
T_e X &
\end{tikzcd}
$$



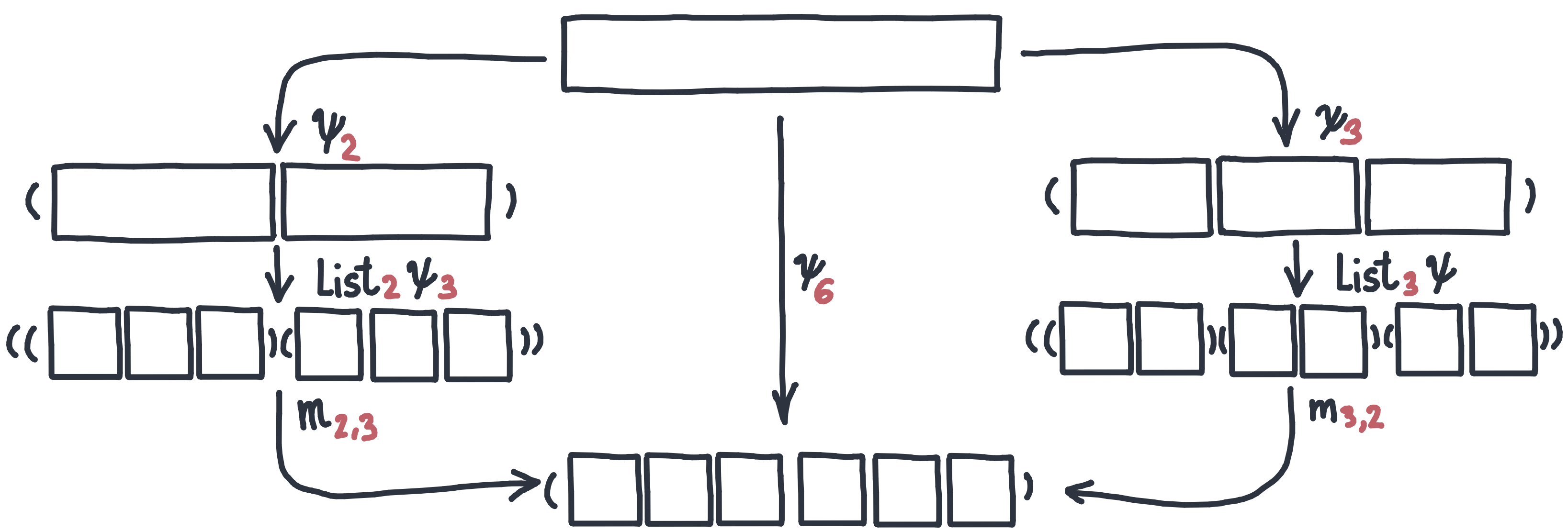


LIST GRADED MONAD

Lists form a graded monad, by naturals with multiplication,

$$\text{flatten} : \mathrm{List}_n \mathrm{List}_m X \longrightarrow \mathrm{List}_{nm} X \,;$$
$$\text{singleton} : X \longrightarrow \mathrm{List}_1 X \,;$$

Rational intervals, $\mathrm{Int} = \{[x,y] \mid x,y \in \mathbb{Q}\}$, form a coalgebra for the graded list mond.
We define $\psi_n : \mathrm{Int} \to \mathrm{List}_n(\mathrm{Int})$ by $\psi_n[x,y] = [[z_0,z_1],\dots,[z_{n-1},z_n]]$ with $z_i = x + i\,(y-x)/n$.



RECOVERING USUAL EXAMPLES

In which sense are we generalizing $(\mathbb{N},+,0)$?

PROPOSITION. Coalgebras for an endofunctor F are the same thing as $(\mathbb{N},+,0)$-graded coalgebras for $(\mathbb{N},+,0)$-graded monad $F^{\circ n}$ given by n-fold functor composition.

What about Lawvere dynamical systems?

PROPOSITION. Lawvere dynamical systems, homomorphisms $(M,\cdot,e) \to (\mathrm{Kl}T(X;X), ⨾, \mathrm{id}_X)$ for a monad T, are the same thing as the coalgebras for the trivially $(M,\cdot,e)$-graded monad $T_x = T$.

Brownian Motion

Brownian motion forms a graded coalgebra for the subGiry monad on standard Borel spaces.

$$\beta_t : \mathbb{R}^n \longrightarrow \mathcal{G}\mathbb{R}^n$$
$$\beta_t(x) = \mathrm{Normal}(\mu = x \,;\, \sigma = t).$$

The coalgebra only contains the position: Brownian motion is memoryless in this sense.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\beta_s"] \arrow[d, "\beta_{s+t}"'] \arrow[dr, phantom, "="] & \mathcal{G}\mathbb{R} \arrow[d, "\mathcal{G}\beta_t"] \\
\mathcal{G}\mathbb{R} & \mathcal{G}\mathcal{G}\mathbb{R} \arrow[l, "\mu"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\beta_s"] \arrow[d, "\beta_s"'] \arrow[dr, phantom, "="] & \mathcal{G}\mathbb{R} \arrow[dl, no head] \\
\mathcal{G}\mathbb{R} &
\end{tikzcd}$$

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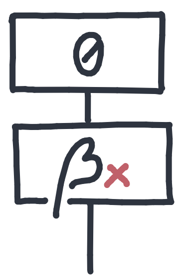

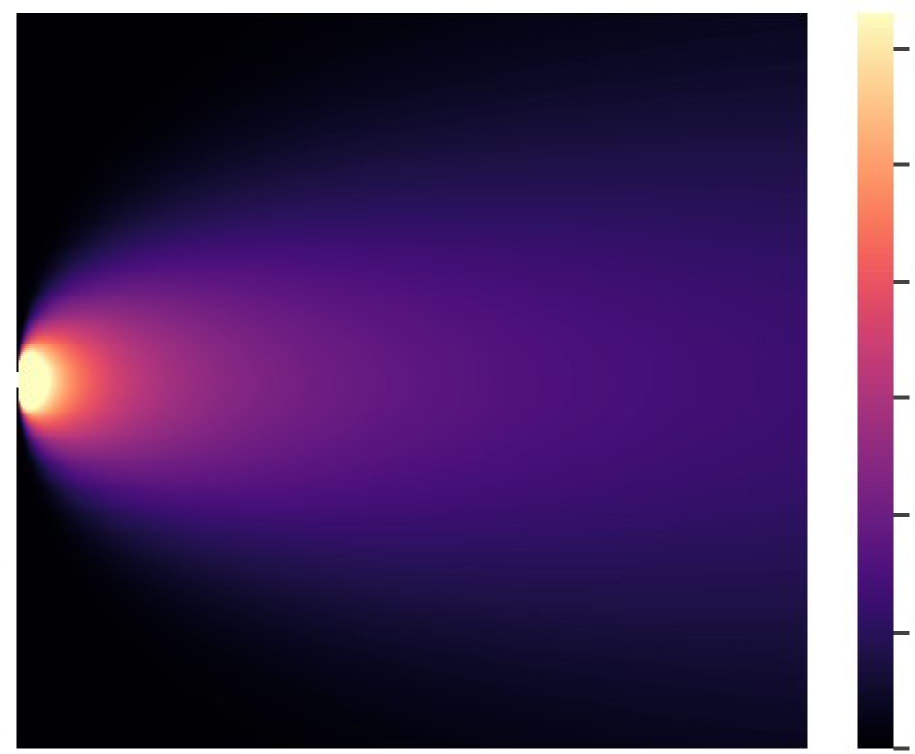

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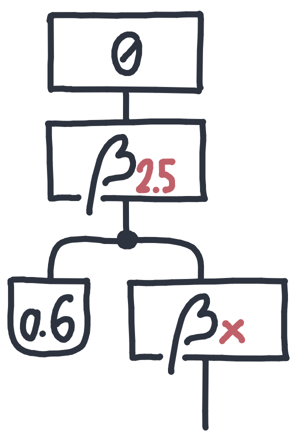

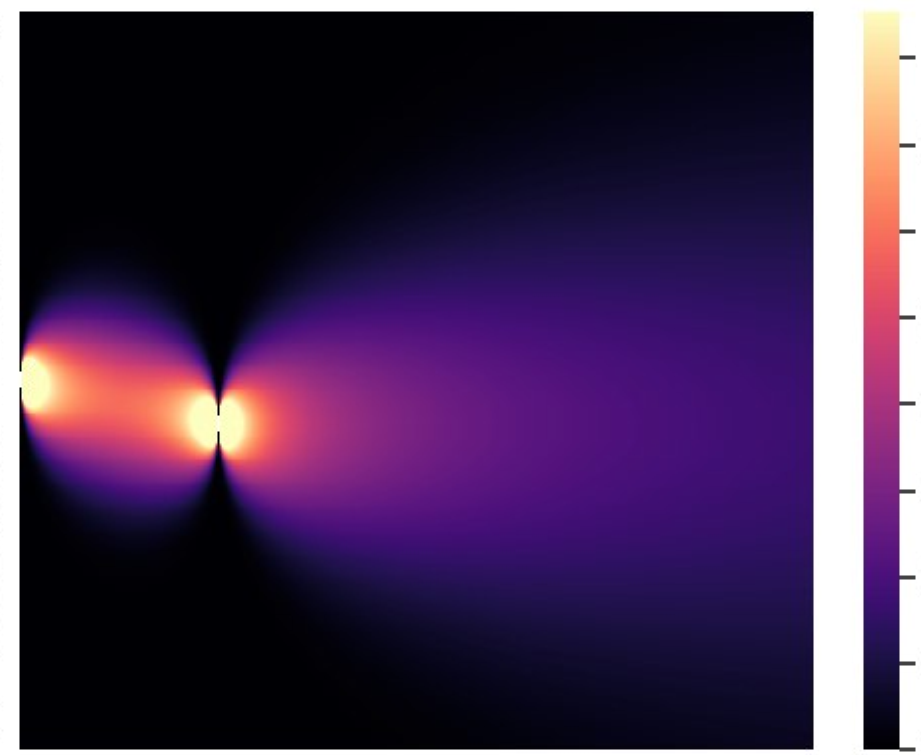

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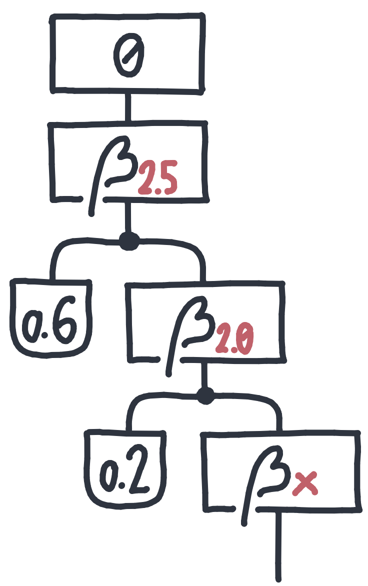

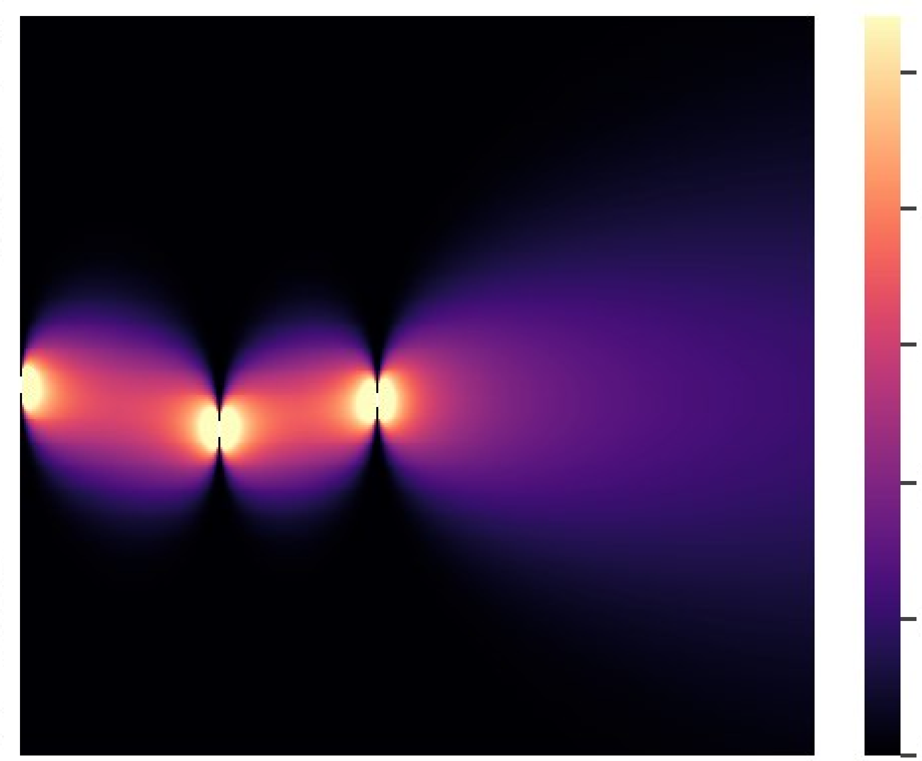

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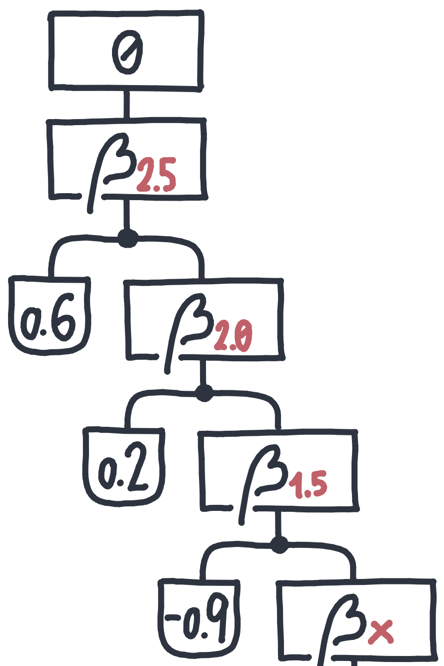
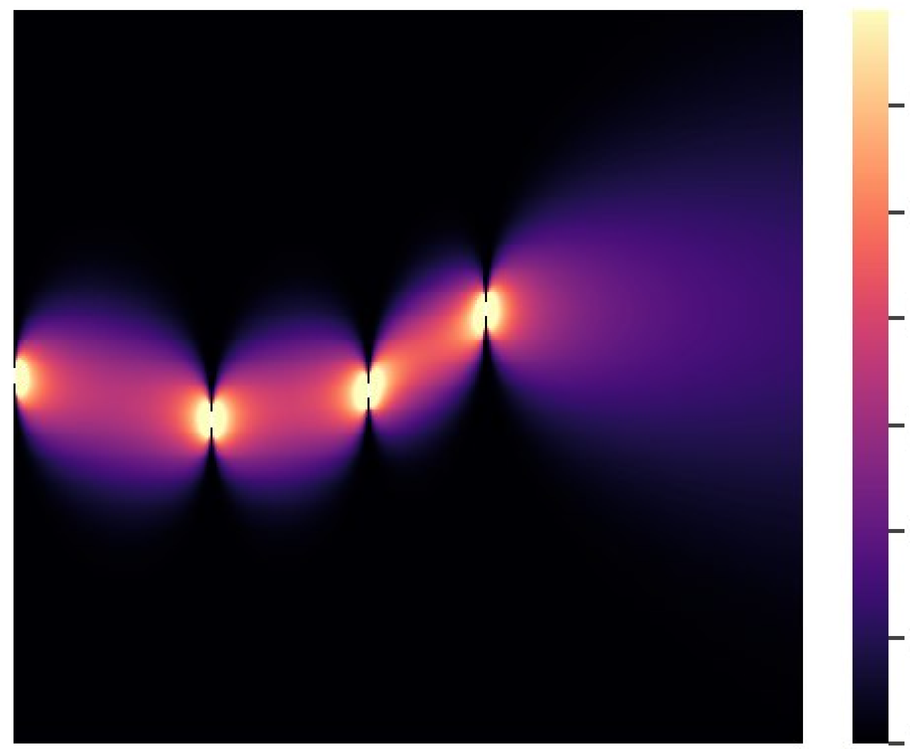


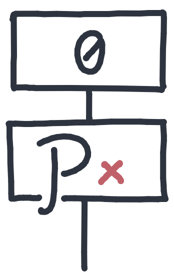


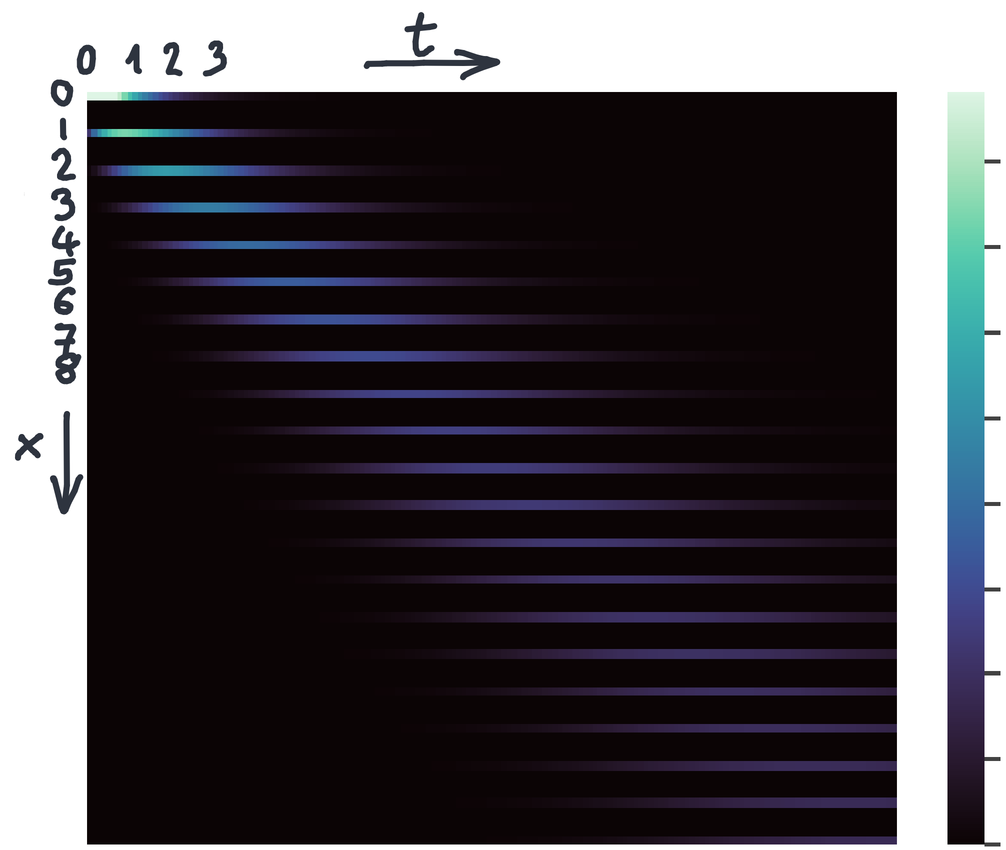
c.f. LazyPPL. Dash, Kaddar, Paquet, Staton. c.f. Stein, Staton. Di Lavore, Román

Poisson Events

The probability of a given number of events with a constant rate.

$$\mathcal{P}_t : \mathbb{R}^n \longrightarrow G\mathbb{R}^n$$
$$\mathcal{P}_t(x) = \text{poisson}(\lambda = t, x).$$





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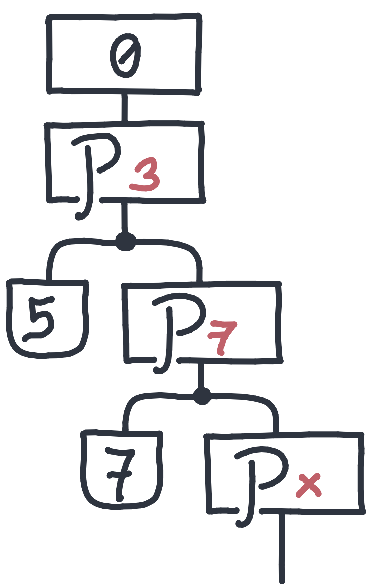

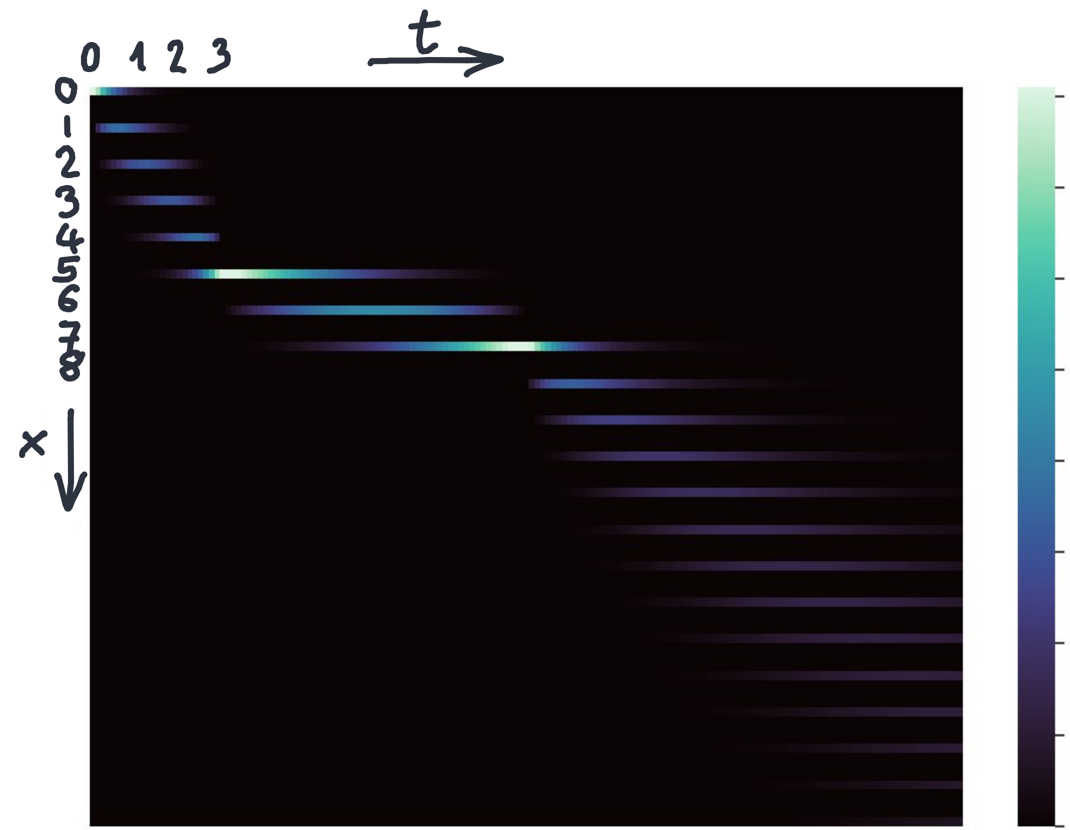

CONCLUSIONS.

- Developing framework for continuous coalgebra.
- More complex stochastic examples run into numerical limitations.
- Numerical methods feel ad-hoc, but we should be able to work symbolically.
- Final coalgebras over a comonad are NOT fixpoints. Same with graded monads.

END

Possibilistic Example

Take a position and some "fuel" that can be used to move at fixed speed.

$$\alpha_t : \mathbb{R}\times\mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R}\times\mathbb{R})$$

$$\alpha_t(x,f) = \{ (x',f') \mid |x'-x| \overset{\text{speed restriction}}{\le} t \text{ and } |x'-x| \overset{\text{fuel}}{\le} f-f' \}$$

Axioms follow from basic triangular inequalities.