# A Canonical Algebra of Open Transition Systems.

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# PART 1:

## SPANS OF GRAPHS

# Span(Graph), Algebra of Open Transition Systems

Many systems do not communicate by I/O message passing, but by synchronization on a common boundary.

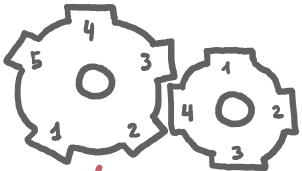
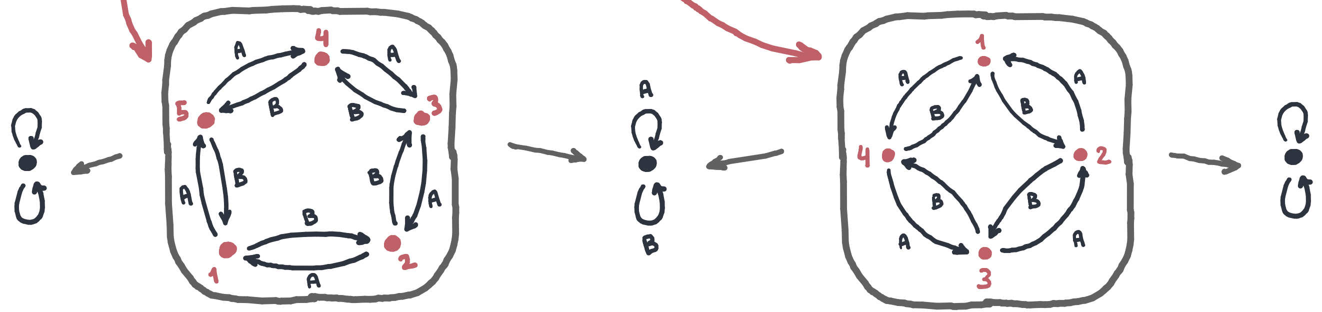



FIG 1. Two gears move together.

How to model this situation? We use spans of graphs.



# Span(Graph), Algebra of Open Transition Systems

**DEFINITION.** A span of functions from a set $A$ to a set $B$ is a pair of functions with a common domain and codomain on $A$ and $B$.

$$A \xleftarrow{f} X \xrightarrow{g} B$$

Something happens in the apex that gets reflected on the boundaries.

Spans can be composed by pullback: synchronization along a boundary.

$$\left(A \xleftarrow{f} X \xrightarrow{g} B\right) ⨾ \left(B \xleftarrow{h} Y \xrightarrow{k} C\right) =$$

$$\left(A \xleftarrow{f} X \times_B Y \xrightarrow{k} C\right)$$

where $X \times_B Y = \{(x,y) \mid x \in X, y \in Y, g(x) = h(y)\}$.

# Span(Graph), Algebra of Open Transition Systems

Each gear can be modelled as a finite state machine graph: states are vertices, and transitions are edges.

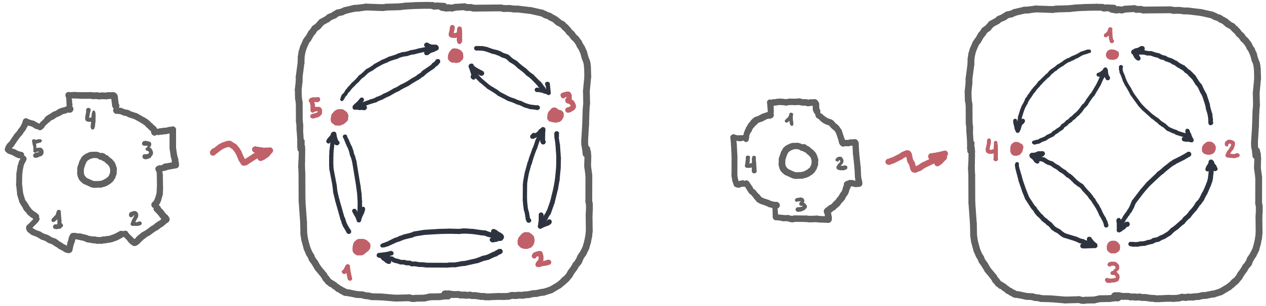



Fig 1. Two gears with 5 states and 4 states, respectively.

The next step is to give interfaces to the different components.

# Span(Graph), Algebra of Open Transition Systems

Each gear can be modelled as a finite state machine graph: states are vertices, and transitions are edges. Each graph has a left and a right interface labelling the transitions.

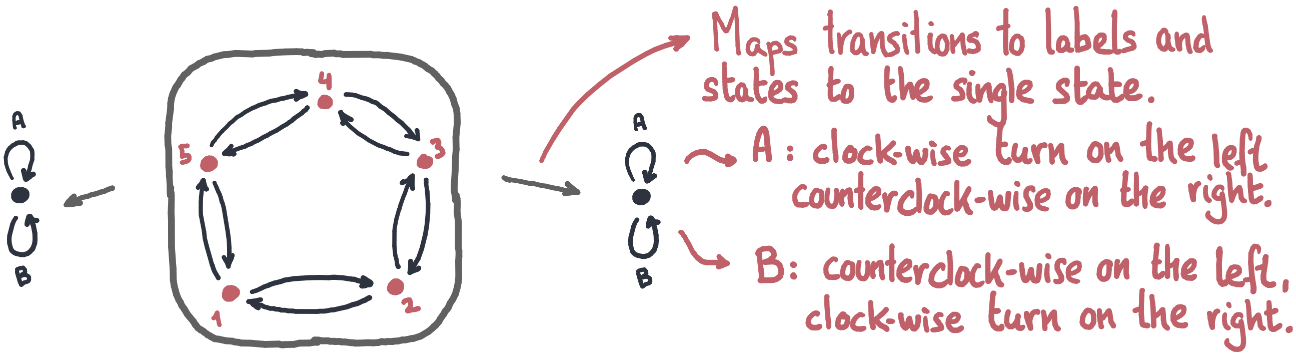
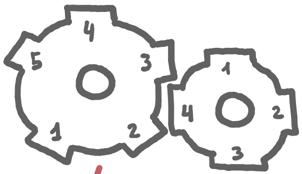


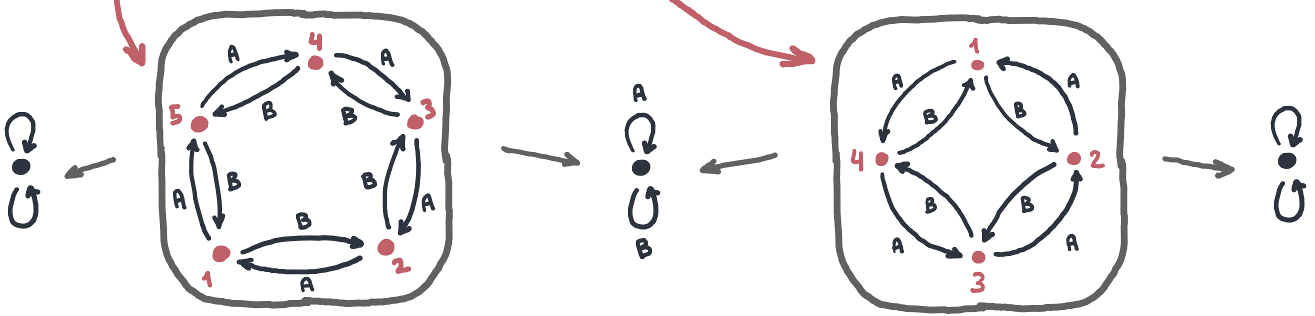

Fig 1. Gear with left and right interfaces.

# Span(Graph), Algebra of Open Transition Systems

Synchronous composition along a boundary is achieved by only allowing transitions whose image coincide on the interface.




Synchronous composition of the two gears in Span(Graph).



# Span(Graph), Algebra of Open Transition Systems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# Span(Graph), Algebra of Open Transition Systems

Span(Graph) is a good algebra of transition systems:

1. Describes compositional, stateful transition systems.
2. Synchronization is plain composition.
3. Transition systems are encoded as graphs.
4. Boundaries may be single vertex graphs without losing any expressivity: Span(Graph)*.

**Problem:** how to justify Span(Graph)*? how is it canonical?

> "It is too easy to cook yet another variant process algebra."
>
> – S. Abramsky

# Span(Graph), Algebra of Open Transition Systems

## Plan for this talk:

1. Show that Span(Graph)* arises from adding **state** to the category of spans.
2. Show that adding state is the same as freely adding **feedback**.
3. Thus, Span(Graph)* is **canonically** the free category with feedback over spans.

"Stateful synchronization is spans of graphs."

4. Conclusions and **variants** of feedback.

# PART 2:

## STATEFUL MORPHISMS

# THE ST(•) CONSTRUCTION

**DEFINITION.** Let $(\mathbb{C}, \otimes, I)$ form a symmetric monoidal category. The category $\mathrm{St}(\mathbb{C})$ has pairs $(S, \varphi)$ as morphisms, where $S \in \mathrm{obj}\,\mathbb{C}$ is the "state space" and $\varphi : S \otimes A \to S \otimes B$ is a "stateful morphism" from A to B.

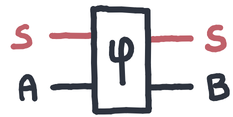



FIG 1. Stateful morphism.

Two stateful morphisms

$$\varphi : S \otimes A \to S \otimes B \quad \text{and} \quad \psi : T \otimes A \to T \otimes B$$

are equal if there exists an isomorphism of state spaces $\gamma : S \cong T$ such that $(\gamma^{-1} \otimes id_A) \circ \varphi \circ (\gamma \otimes id_B) = \psi$.

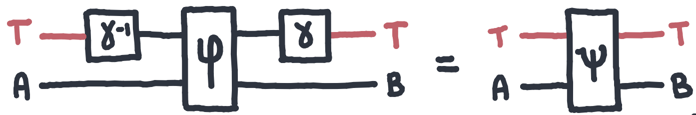



FIG 2. Equivalence relation on stateful morphisms.

# The St(•) Construction

**PROPOSITION** (Katis, Sabadini, Walters, 1997). Stateful morphisms over $\mathbb{C}, \otimes$ form a symmetric monoidal category $\mathrm{St}(\mathbb{C})$, with the same objects as $\mathbb{C}$.

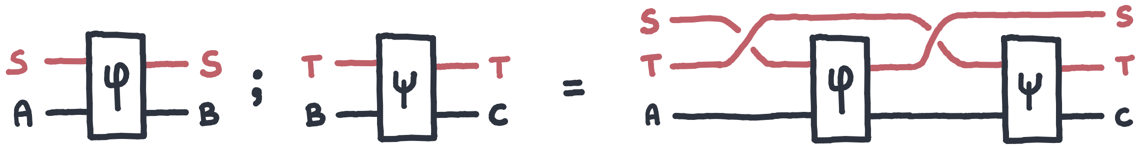



FIG 1. Sequential composition of stateful morphisms.

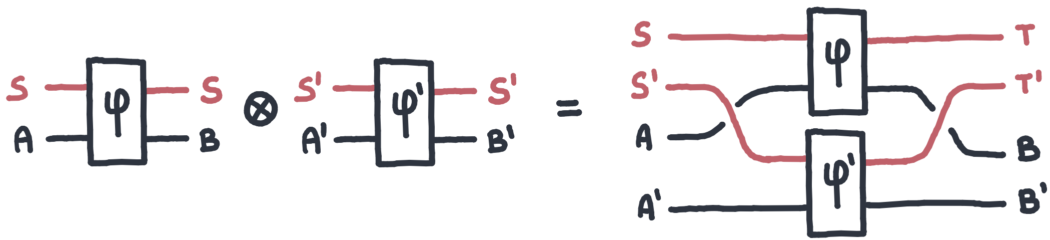



FIG 2. Parallel composition of stateful morphisms.

# The $\mathsf{St}(\bullet)$ Construction

Stateful morphisms over well-known monoidal categories form some well-known categories of transition systems.

- Let $\mathbb{C} = \mathbf{Set}^{\times}$. Stateful functions $S \times A \to S \times B$ are Mealy transition systems, $\mathsf{St}(\mathbf{Set}^{\times}) \cong \mathbf{Mealy}$.
- Let $\mathbb{C} = \mathbf{Set}^{+}$. Stateful functions $S + A \to S + B$ are Elgot transition systems, $\mathsf{St}(\mathbf{Set}^{+}) \cong \mathbf{Elgot}$.
- Let $\mathbb{C} = \mathbf{Rel}$. Stateful relations $S \times A \nrightarrow S \times B$ are Non-deterministic transition systems, $\mathsf{St}(\mathbf{Rel}) \cong \mathbf{Ndts}$.
- Let $\mathbb{C} = \mathbf{Stoch}$. Stateful stochastic functions $S \times A \to D(S \times B)$ are probabilistic transition systems, $\mathsf{St}(\mathbf{Stoch}) \cong \mathbf{Pts}$.

# The St(•) Construction

Consider the monoidal category of spans of functions with the cartesian product, $\mathsf{Span}(\mathsf{Set})^{\times}$. A morphism $A \nrightarrow B$ is a span

$$A \xleftarrow{a} E \xrightarrow{b} B.$$

What is a stateful span $S \times A \nrightarrow S \times B$? It is a span of sets, but it can also be seen as a span of graphs:

$$S \times A \xleftarrow{s \times a} E \xrightarrow{t \times b} S \times B,$$ or also

$$\begin{tikzcd}
A \arrow[d, bend right, "!"'] \arrow[d, bend left, "!"] & E \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, bend right, "s"'] \arrow[d, bend left, "t"] & B \arrow[d, bend right, "!"'] \arrow[d, bend left, "!"] \\
1 & S \arrow[l, "!"] \arrow[r, "!"'] & 1
\end{tikzcd}$$

**THEOREM** (DGRSS, 2020). There exists a monoidal isomorphism

$$\mathsf{St}(\mathsf{Span}(\mathsf{Set})^{\times}) \cong \mathsf{Span}(\mathsf{Graph})^{*}.$$

"Stateful synchronization is spans of graphs."

# The St(•) Construction

Feedback and state are closely interrelated concepts. A remarkable fact from electronic circuit design is how data-storing components can be constructed from stateless components and feedback.

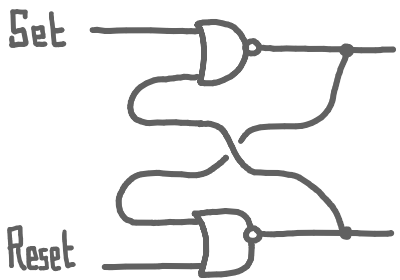



FIG 1. The Set/Reset NOR latch.

Activating any of the inputs sets the output permanently.

We will describe a construction that adds "state" to a monoidal category. The same construction will add "feedback".

# Part 3:

## Feedback

# CATEGORIES WITH FEEDBACK

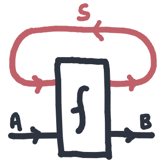



FIG 1. Notation.

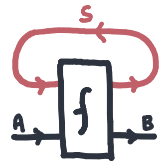
**DEFINITION** (Katis, Sabadini, Walters, 1997). A category with feedback is a symmetric monoidal category endowed with an operator

$$\mathsf{fbk}_S : \hom(S \otimes A, S \otimes B) \to \hom(A, B),$$ such that

## AXIOMS.

1. $u \circ \mathsf{fbk}_S(f) \circ v = \mathsf{fbk}_S((u \otimes id) \circ f \circ (v \otimes id))$,
2. $\mathsf{fbk}_I(f) = f$,
3. $\mathsf{fbk}_S(\mathsf{fbk}_S(f)) = \mathsf{fbk}_{S \otimes T}(f)$,
4. $\mathsf{fbk}_S(f) \otimes g = \mathsf{fbk}_S(f \otimes g)$,
5. $\mathsf{fbk}_S(f \circ (h \otimes id)) = \mathsf{fbk}_T((h \otimes id) \circ f)$,
   when $h$ is an isomorphism.

# Categories with Feedback

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## Axioms.

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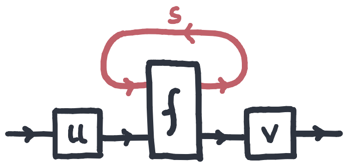
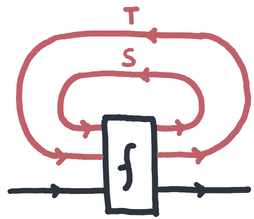
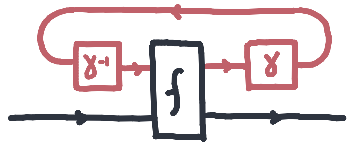



3.




5.

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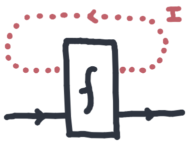


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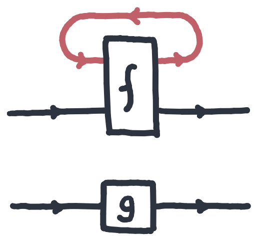
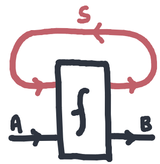


# CATEGORIES WITH FEEDBACK

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## AXIOMS.

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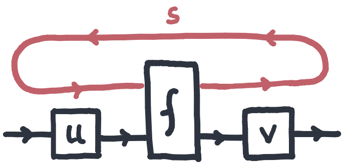


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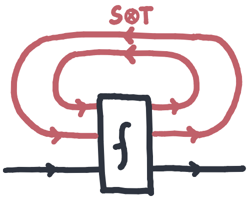


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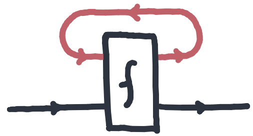


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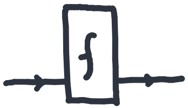


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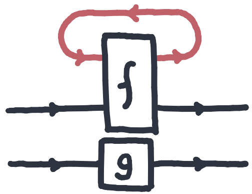
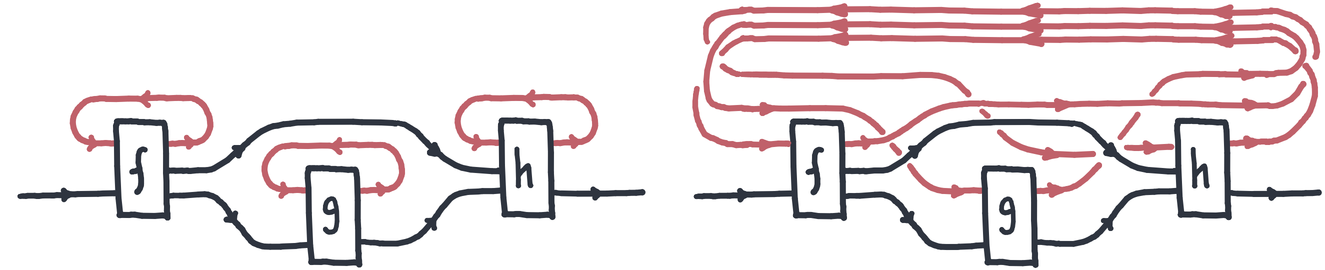


# Categories with Feedback

**THEOREM** (Katis, Sabadini, Walters, 2002). Diagrams in a category with feedback have a normal form with a single feedback loop.




**COROLLARY** (DGRSS, 2020). The category $\mathsf{St}(\mathbb{C})$ of stateful morphisms is the free category with feedback over $\mathbb{C}$. There exists an adjunction

$$\begin{tikzcd} \mathsf{SymMon} \arrow[r, bend left, "\mathsf{St}"] \arrow[r, phantom, "\perp"] & \mathsf{Feedback} \arrow[l, bend left, "\mathsf{U}"] \end{tikzcd}$$

$$\mathsf{St}(\mathsf{Span}(\mathsf{Set})^{\times}) \cong \mathsf{Span}(\mathsf{Graph})^{*}.$$

"Feedback & synchronization is spans of graphs."

# Part 4:

## Generalizing $St(\cdot)$

# Conclusion

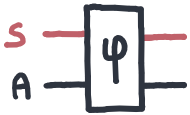
- **Main Result:** a monoidal isomorphism $\mathsf{St}(\mathsf{Span}(\mathsf{Set})^{\times}) \cong \mathsf{Span}(\mathsf{Graph})^{*}$.
- **Consequence:** a universal property for $\mathsf{Span}(\mathsf{Graph})^{*}$.
- **More Generally:** the $\mathsf{St}(\bullet)$ construction from the literature is the free category with feedback.
- Categories with **Feedback** are a weakening of traces that still have a normal form theorem.
- The $\mathsf{St}(\bullet)$ construction can be generalized to other variants of feedback.

## Future Work.

- Variants of feedback, how to categorically describe them.
- Applications to stream-based programming.

# GENERALIZING THE ST(•) CONSTRUCTION

What is the most general form of state? What is the most general form of feedback?




S ranges over arbitrary monoidal categories.

S acts on A and B with an arbitrary monoidal action.

FIG 1. Stateful morphism.

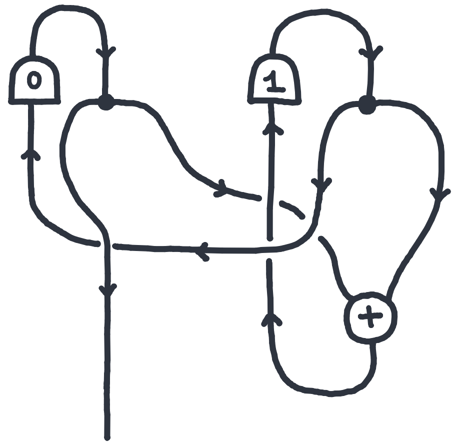



FIG 1. Fibonacci, using initialized feedback.

We recover multiple variants of feedback.

1. Guarded feedback.
2. Initialized feedback.
3. Delayed feedback.

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# CATEGORIES WITH FEEDBACK

What are the differences with traced monoidal categories?

1. Only isomorphisms can be slided.


2. The yanking equation does not hold.


More technically,

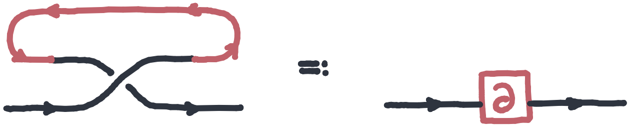
- feedback is weaker than trace, and
- feedback coincides with guarded trace in compact closed categories.

# CATEGORIES WITH FEEDBACK

What are the differences with traced monoidal categories?

1. Only isomorphisms can be slided.


2. The yanking equation does not hold. There is a delay.

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