Streams in Monoidal Categories.

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Main Result

"From a theory of processes to a theory of streams."

Monoidal categories encode theories of processes.

Stream-based programming in monoidal categories.

Construction. Let $\mathbb{C}$ be a symmetric monoidal category. $\textsc{Stream}(\mathbb{C})$ is a symmetric monoidal category with feedback and an id-on-objects functor of the form $\mathbb{C}^{\mathbb{N}} \to \textsc{Stream}(\mathbb{C})$.

Part 0:

Monoidal Categories.

Monoidal Categories

Monoidal categories encode theories of processes that compose sequentially and in parallel. Each theory of processes gives rise to a symmetric monoidal category. Monoidal categories come with a practical graphical calculus.

EXAMPLES:

- Sets and functions. (Set, $\times$)
- Relational queries. (Rel, $\times$)
- Stochastic functions (Stoch, $\times$)
- Linear operators. (Vect, $\otimes$)
- Quantum processes. (Hilb, $\otimes$)

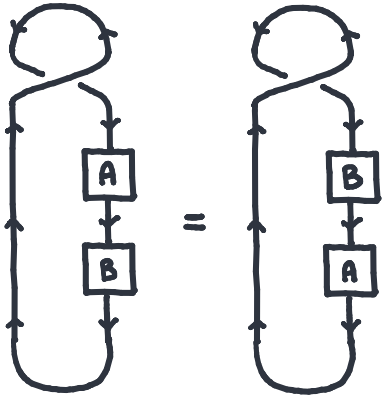

FIG 1. Cyclicity of the trace operator.

$$tr(AB) = tr(BA)$$

Part 1:
Stream-Based Programming

Stream-based Programming

In causal stream-based programming, every declaration is a stream that depends on its previous values.

Lucid

```
n = 0 fby [n+1]
sum x = x + [0 fby [sum x]]
fib = 0 fby [1 fby [ fib + next fib ]]
```

A stream function from $X = X_0, X_1, \ldots$ to $Y = Y_0, Y_1, \ldots$ produces, at each time $t = 0, 1, \ldots$, an output Y_t from all previous inputs $X_0, X_1, \ldots, X_t$.

Stream-based Programming

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```
# Lucid                                  "followed by"
n = 0 fby [n+1]                    ⟶  0,1,2,3,...
sum x = x + [0 fby [sum x]]        ⟶  sum n = 0,1,3,6,10,...
fib = 0 fby [1 fby [fib + next fib]]  ⟶  fib = 0,1,1,2,3,5,8,13...
```

A stream function from $X = X_0, X_1, \ldots$ to $Y = Y_0, Y_1, \ldots$ produces, at each time $t = 0, 1, \ldots$, an output Y_t from all previous inputs $X_0, X_1, \ldots, X_t$.

How to bring streams to any monoidal category?

Stream-based Programming

Plan for this talk.

1. Construct a theory of streams over an arbitrary theory of processes.
2. Describe the theory of streams with a canonical coinductive definition.
3. Example: Recover the classical theory of "cartesian" streams.
4. Example: Obtain a notion of stochastic stream.
5. Feedback syntax for streams.

Part 2:

Stream Transformers

Stream Transformers

Construction (Sprunger, Katsumata, 2019). A stateful morphism sequence is a family of morphisms that, at each instant in time $t = 0, 1, 2, \ldots$, read from a memory M_{t-1}, take an input X_t, produce an output Y_t, and write to a new memory M_t.

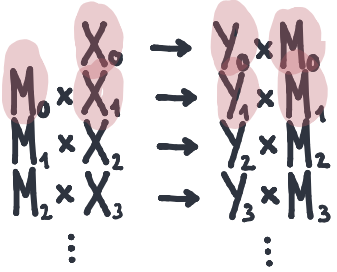

Fig 1. Stateful sequence.

Questions.

- Can we generalize to monoidal categories?
- How to justify this construction?
- Technical point: when are two sequences equal?

Stream Transformers

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$$
\begin{array}{ccc}
X_0 & \to & Y_0 \times M_0 \\
M_0 \times X_1 & \to & Y_1 \times M_1 \\
M_1 \times X_2 & \to & Y_2 \times M_2 \\
M_2 \times X_3 & \to & Y_3 \times M_3 \\
\vdots & & \vdots
\end{array}
$$

Fig 1. Stateful sequence.

Questions.

- Can we generalize to monoidal categories?
  Yes (Román, 2020).
- How to justify this construction?
  It's a final coalgebra (DFR, 2021).
- Technical point: when are two sequences equal?
  Dinaturality until stage n (DFR, 2021).

Stream Transformers

Construction (DFR, 21). Let $\mathbb{C}, \otimes$ be a symmetric monoidal category. The symmetric monoidal category $\mathrm{Stream}_{\mathbb{C},\otimes}$ has the same objects as $\mathbb{C}^{\mathbb{N}}$ and stateful morphism sequences as morphisms,

$$\mathrm{Stream}(X;Y) = \lim_n \int^{M_0,\ldots,M_n} \prod_{i=0}^{n} \hom(M_{i-1} \otimes X_i, Y_i \otimes M_i).$$

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Limit $n \to \infty$. For each n, describe the sequence until n.

Coend. Dinatural existential quantifier. Choose memory channels.

Input memory. Input. Output. Output memory.

Stateful sequence.

Stream Transformers

Streams have a coinductive definition. $\mathrm{Str}(A_0, A_1, \ldots) = A_0 \times \mathrm{Str}(A_1, A_2, \ldots)$.

> "A stream with types $\mathbf{A} = A_0, A_1, A_2, \ldots$ is an element of A_0 together with a stream with types $A_1, A_2, \ldots$."

Stream transformers can be also given a coinductive definition.

> "A stream transformer from $\mathbf{X} = X_0, X_1, \ldots$ to $\mathbf{Y} = Y_0, Y_1, \ldots$ is a process from X_0 to Y_0 communicating along a memory with a stream transducer from $X_1, X_2, \ldots$ to $Y_1, Y_2, \ldots$."

$$\mathrm{Stream}(\mathbf{X};\mathbf{Y}) = \int^{M} \hom(X_0, Y_0 \otimes M) \times \mathrm{Stream}(M \otimes X_1, X_2, \ldots; Y_1, Y_2, \ldots).$$

Memory channel

Coend: natural existential quantifier

Process from the base category

Another stream

Stream Transformers

$$\mathrm{Stream}(X;Y) = \int^{M} \hom(X_0, Y_0 \otimes M) \times \mathrm{Stream}(M \otimes X_1, X_2, \ldots; Y_1, Y_2, \ldots).$$

Memory channel

Coend: natural existential quantifier

Process from the base category

Another stream

This construction:

1. Generalizes the stateful sequences of Sprunger & Katsumata.
2. Coinductive definition.
3. Provides a natural notion of stream equivalence.
4. Works in arbitrary monoidal categories.

Part 3:

Examples

Cartesian Processes

Definition (Uustalu, Vene, 2008, adapted). A **causal stream function** is a morphism of the coKleisli category of $M : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$.

$$\frac{X \to Y}{\begin{array}{l} X_0 \to Y_0 \\ X_1 \to Y_1 \\ X_2 \to Y_2 \\ X_3 \to Y_3 \\ \vdots \quad \vdots \quad \vdots \end{array}}$$

Memoryless process in $\mathbb{C}^{\mathbb{N}}$.

$$\frac{MX \to Y}{\begin{array}{l} X_0 \to Y_0 \\ X_0 * X_1 \to Y_1 \\ X_0 * X_1 * X_2 \to Y_2 \\ X_0 * X_1 * X_2 * X_3 \to Y_3 \\ \quad \vdots \qquad \vdots \end{array}}$$

Stream transformer in coKleisli(M).

Theorem (DFR, 2021). There is an equivalence of categories

$$\mathbf{Stream}(\mathbf{Set}) \cong \mathbf{coKleisli}(M).$$

Stochastic Processes

Definition (Ross, 1996, adapted). A stochastic process from X to Y is a sequence of functions $f_n : X_0 \times \cdots \times X_n \to D(Y_0 \times \cdots \times Y_n)$ such that f_n is the marginal distribution of f_{n+1} in the first n variables.

$$\begin{array}{ccc} X_0 \times \cdots \times X_{n+1} & \xrightarrow{f_{n+1}} & D(Y_0 \times \cdots \times Y_{n+1}) \\ \pi \downarrow & = & \downarrow D\pi \\ X_0 \times \cdots \times X_n & \xrightarrow{f_n} & D(Y_0 \times \cdots \times Y_n) \end{array}$$

Theorem (DFR, 2021). There is an equivalence of categories

$$\mathsf{Stream}(\mathsf{Stoch}) \cong \mathsf{StochasticProcesses}.$$

Part 4:

Graphical Syntax & Further Work.

Graphical Syntax

Stateful sequences came with a convenient graphical calculus where input/output goes top-bottom and memory goes left to right.

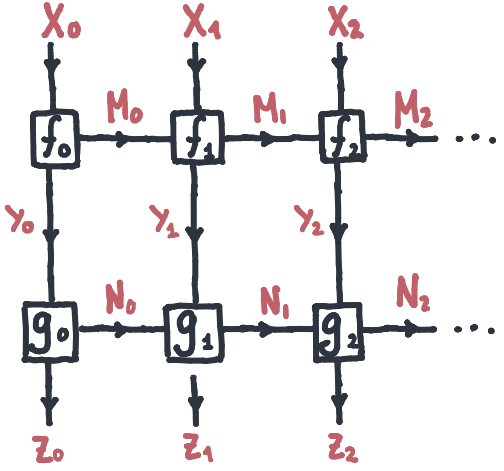


Fig 1. A stateful sequence
$f_n : M_{n-1} \otimes X_n \to Y_n \otimes M_n$
and a stateful sequence
$g_n : N_{n-1} \otimes Y_n \to Z_n \otimes N_n$
compose into a stateful sequence
$(f \fatsemi g)_n : N_{n-1} \otimes M_{n-1} \otimes X_n \to Z_n \otimes M_n \otimes N_n$.

Formalization in terms of "open diagrams": Román 2019, 2020.

Syntax: Categories with Feedback

Can we have a more compact notation? Categories with feedback are a weakening of traces with a graphical calculus in terms of loops.

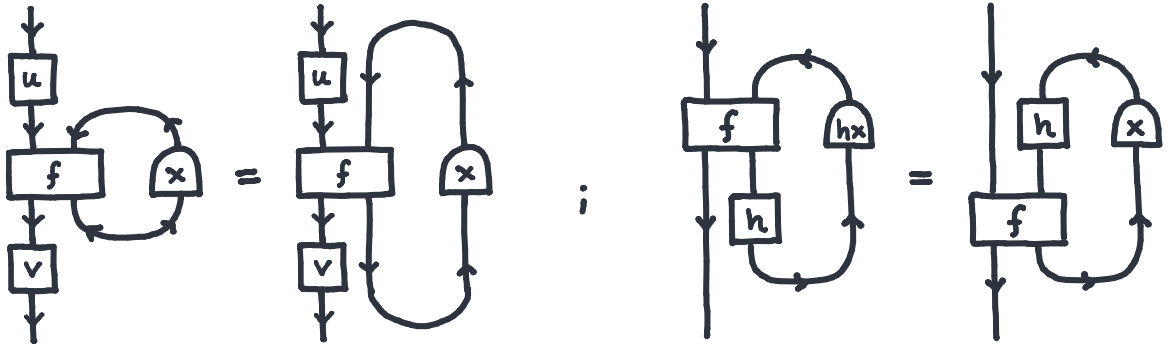
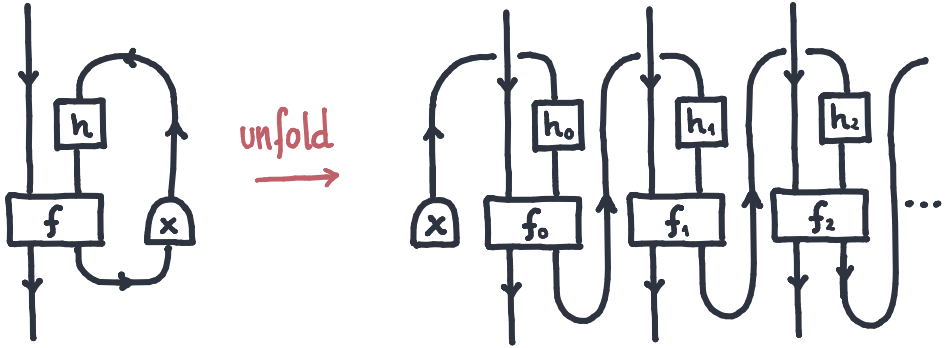


Fig 1. Some axioms for "pointed" or "guarded" feedback.

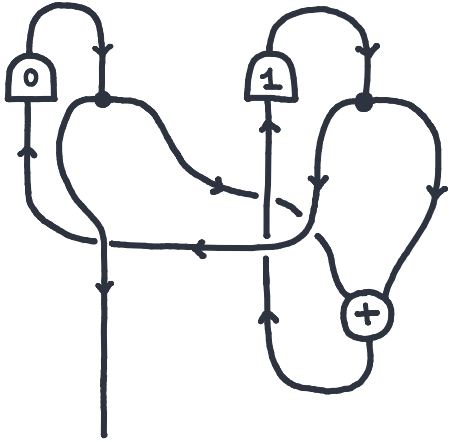
Recipe: use categories with feedback for syntax and stream transformers for semantics.

Syntax: Categories with Feedback

Theorem (DFR, 21). The monoidal category $\mathbf{Stream}_{\mathbb{C},\otimes}$ has feedback. Thus, there exists a feedback-preserving functor from the syntactic category with feedback to $\mathbf{Stream}_{\mathbb{C},\otimes}$.



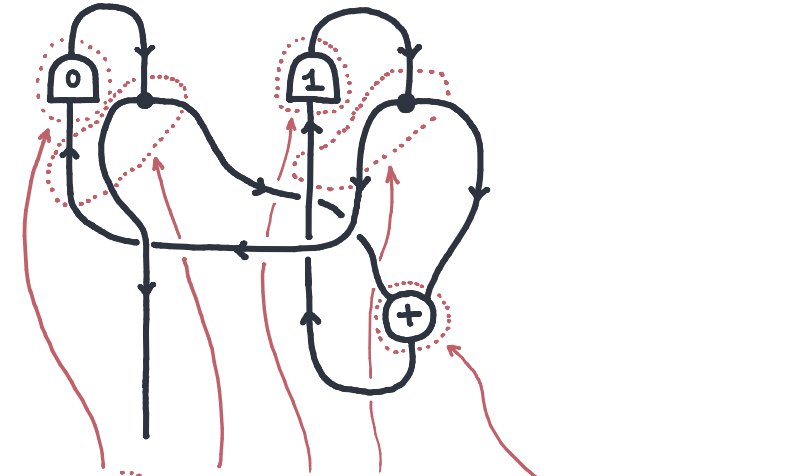
Diagrammatic Stream Programming




```
fib = 0 fby [1 fby [ fib + next fib ]]
```

Lucid

Diagrammatic Stream Programming



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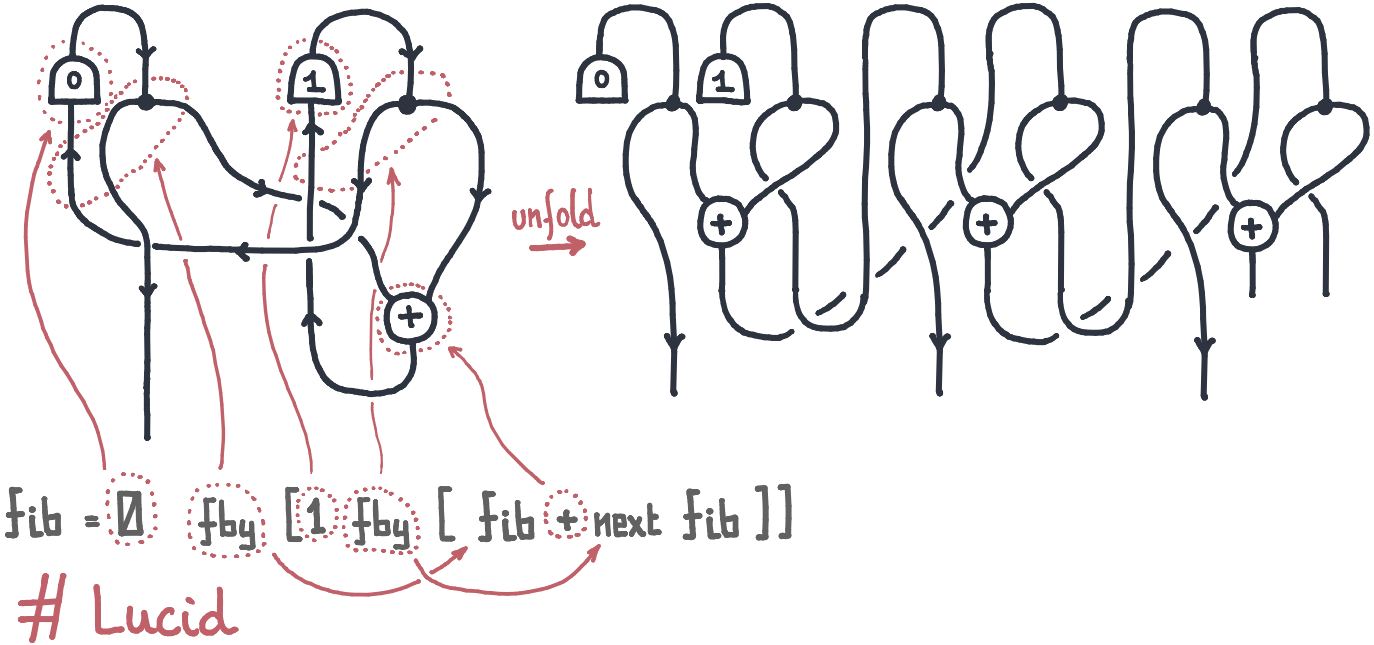

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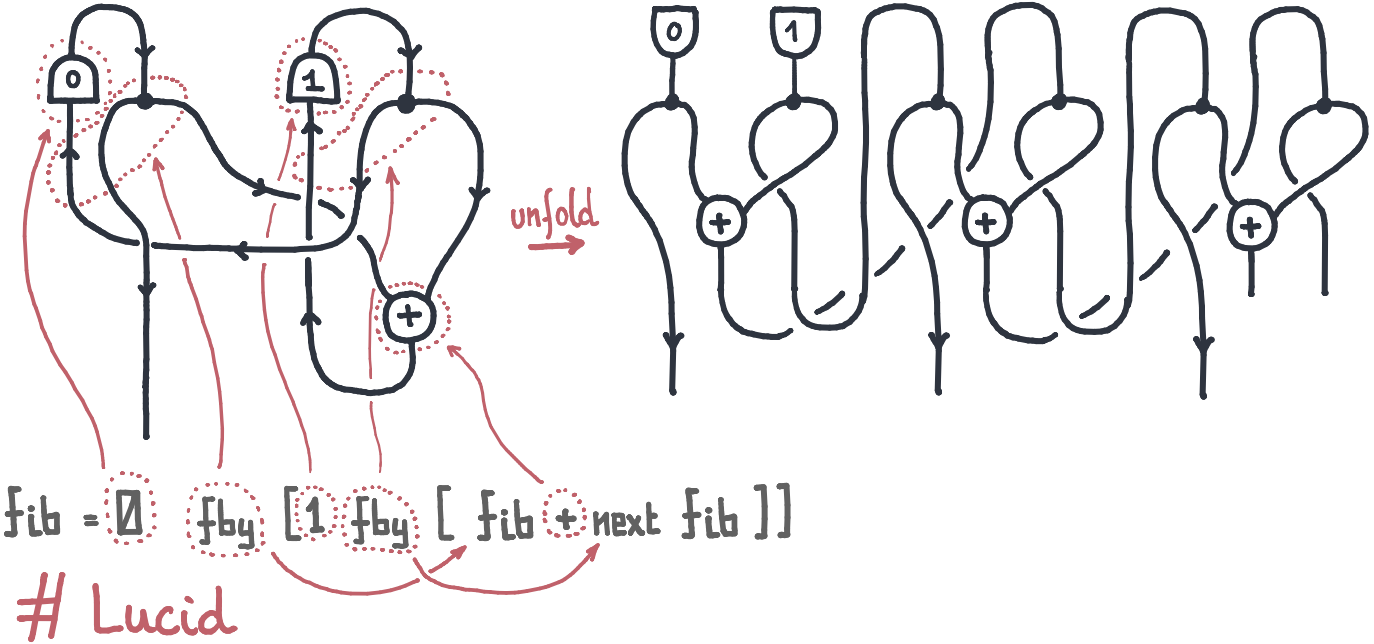

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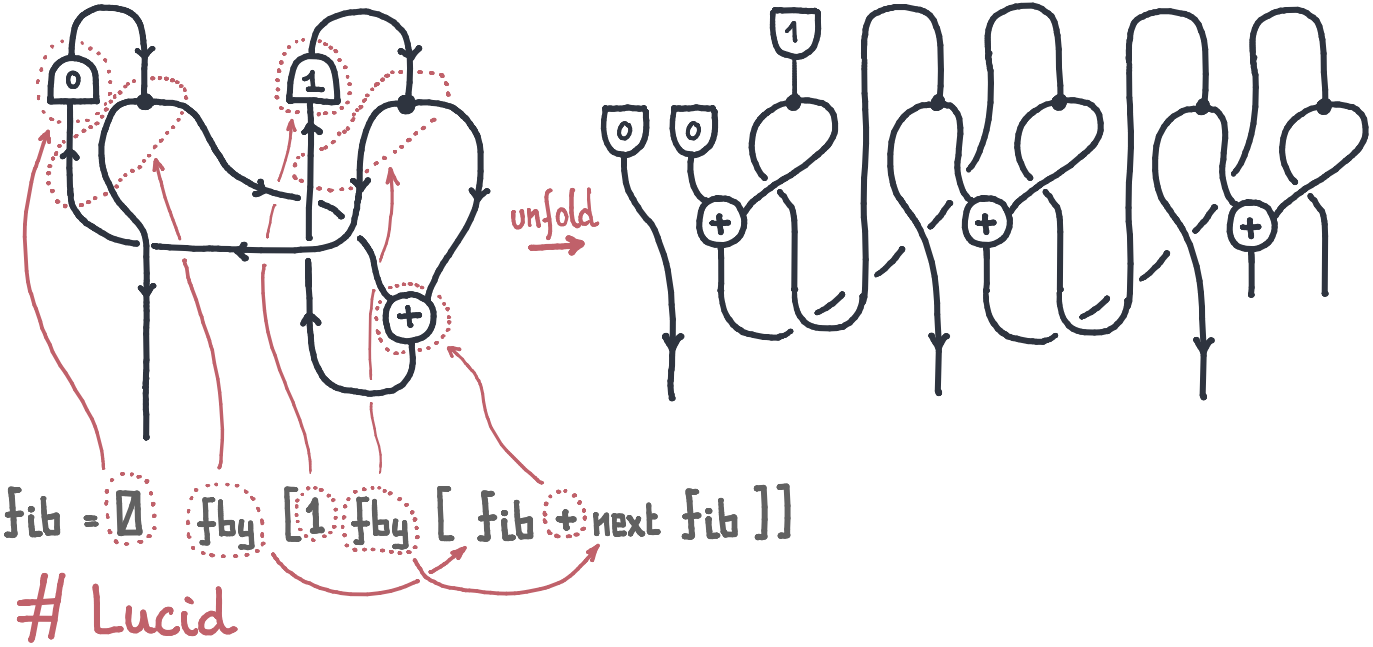

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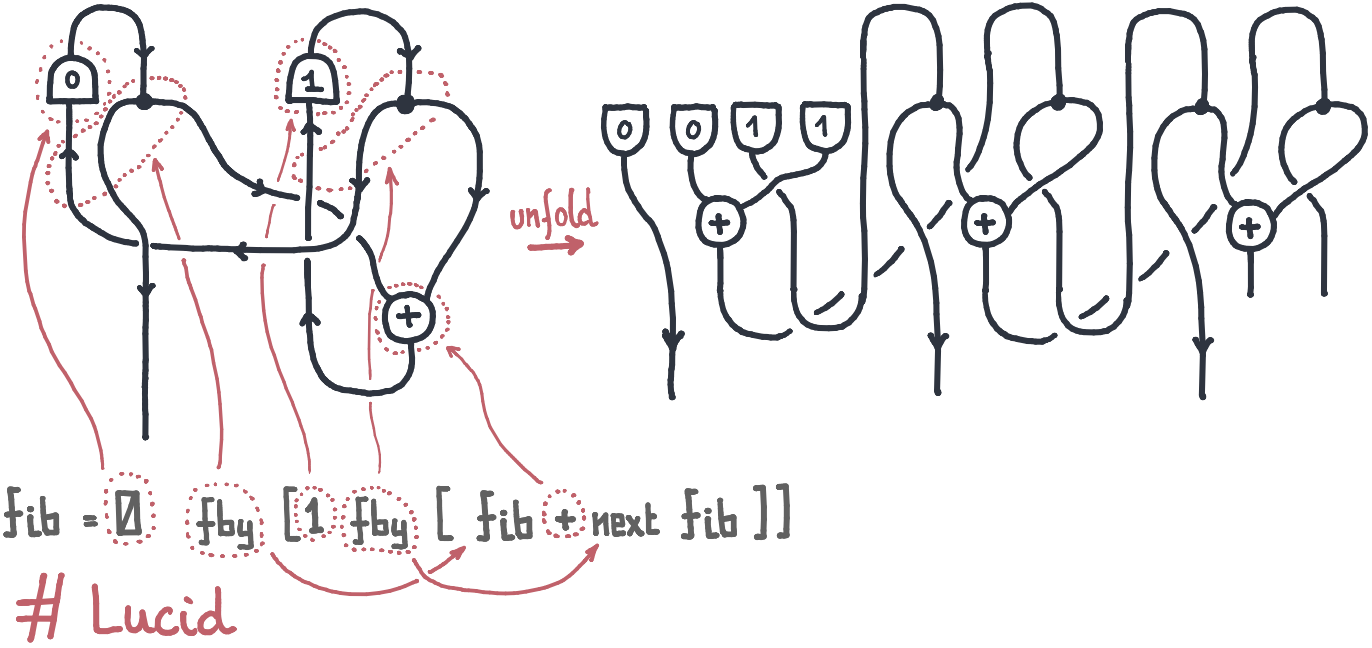

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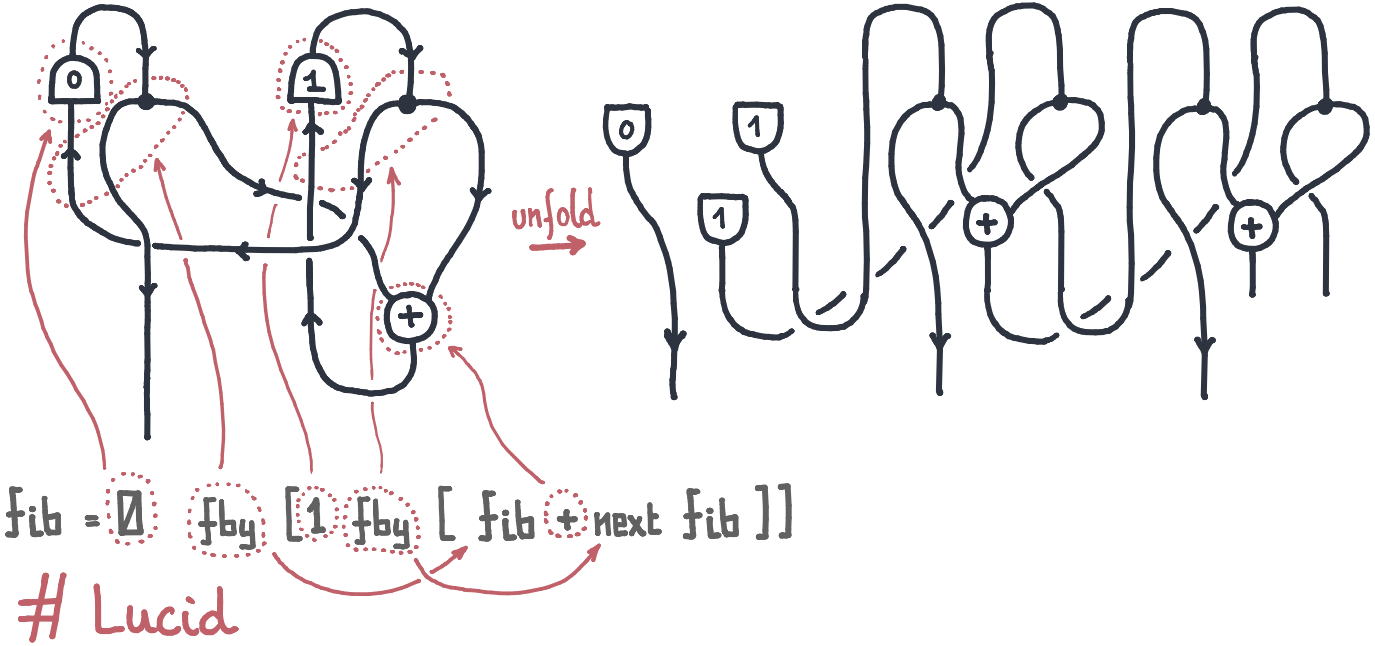

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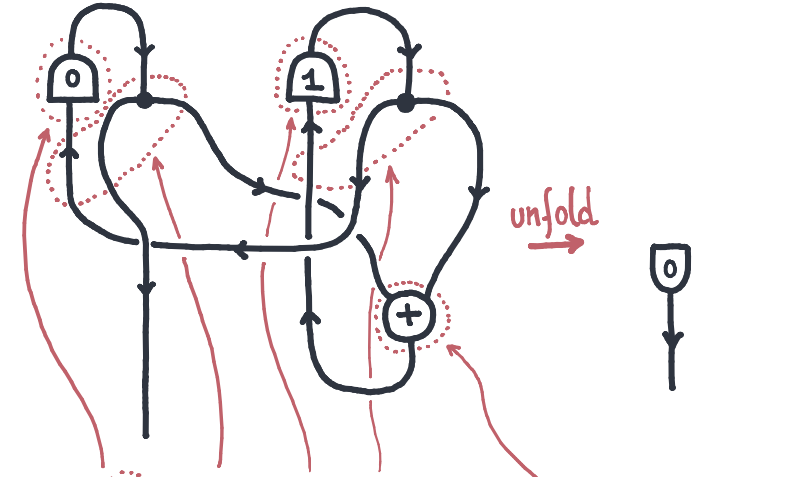
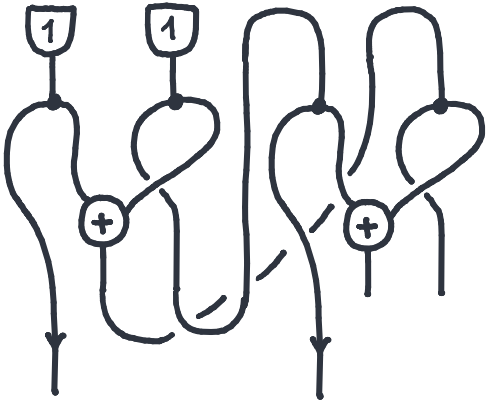
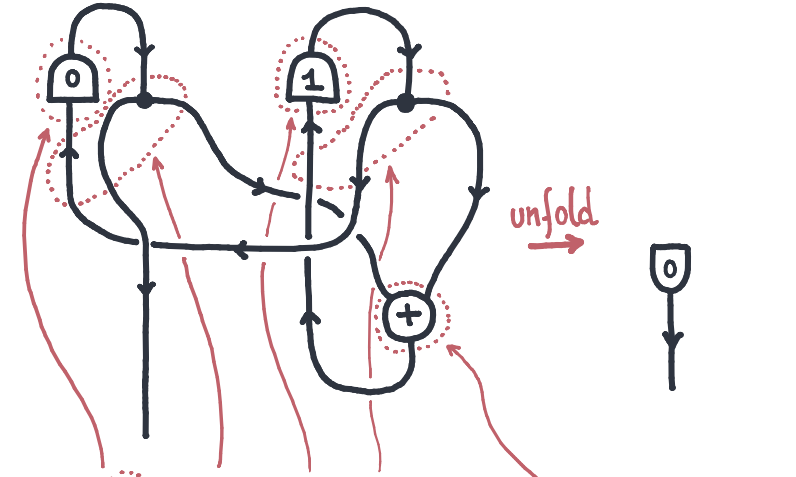


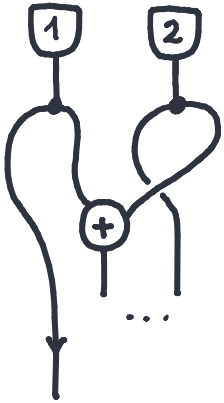


fib = 0 fby [1 fby [fib + next fib]]

Lucid

Diagrammatic Stream Programming





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Lecture Notes in Computer Science, 2006.

PROFUNCTORS

DEFINITION (Communicating composition). Let $P : \mathbb{A}^{op} \times \mathbb{B} \to \mathbf{Set}$ and $Q : \mathbb{B}^{op} \times \mathbb{C} \to \mathbf{Set}$ be profunctors with $\mathbb{B}, \otimes$ a monoidal category. The communicating composition $(P \odot Q) : \mathbb{A}^{op} \times \mathbb{B}^{op} \times \mathbb{B} \times \mathbb{C} \to \mathbf{Set}$ is defined by

$$(P \odot Q)(A, B, B', C) = \int^{M} P(A, B \otimes M) \times Q(M \otimes B', C).$$

REMARK. $\mathbf{Stream}_{\mathbb{C}}$ can be obtained as the greatest fixpoint of

$$\mathbf{Stream} = \hom \odot \mathbf{Stream}.$$

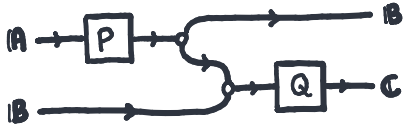

FIG 1. Communicating composition in the graphical calculus of profunctors.

Stream-based Programming

Construction (Uustalu, Vene, 2016). Causal stream functions are morphisms of the cokleisli category of a suitable comonad.

The monoidal category $\mathbb{C}^{\mathbb{N}}$ has families of morphisms of $\mathbb{C}$ indexed by time $t=0,1,\dots$ A "memory" comonad $M : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ can endow each function with a context representing past inputs, $(MX)_n = X_0 \times \cdots \times X_n$.

$X \to Y$

$$
\begin{aligned}
X_0 &\to Y_0 \\
X_1 &\to Y_1 \\
X_2 &\to Y_2 \\
X_3 &\to Y_3 \\
\vdots \quad &\quad \vdots \quad \quad \vdots
\end{aligned}
$$

Memoryless process in $\mathbb{C}^{\mathbb{N}}$!

$MX \to Y$

$$
\begin{aligned}
X_0 &\to Y_0 \\
X_0 \times X_1 &\to Y_1 \\
X_0 \times X_1 \times X_2 &\to Y_2 \\
X_0 \times X_1 \times X_2 \times X_3 &\to Y_3 \\
\vdots \quad &\quad \vdots
\end{aligned}
$$

Stream transformer in cokleisli(M).

Stream-based Programming

Idea (Naïve Mario). Just substitute $(\times)$ by a tensor product $(\otimes)$.

$$MX \to Y$$

Naïve Mario's monoidal stream transformer

$$\begin{array}{l} X_0 \to Y_0 \\ X_0 \otimes X_1 \to Y_1 \\ X_0 \otimes X_1 \otimes X_2 \to Y_2 \\ X_0 \otimes X_1 \otimes X_2 \otimes X_3 \to Y_3 \\ \quad \vdots \qquad\qquad \vdots \end{array}$$

Theorem (DFR, 2021). The functor $M : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ defined by $(MX)_n = X_0 \otimes \cdots \otimes X_n$ has a monoidal comonad structure *if and only if* the category $\mathbb{C}, \otimes$ is cartesian.

So, we want a different generalization to arbitrary monoidal categories.

Stream Transformers

Streams have a coinductive definition. $\mathrm{Str}(A_0, A_1, \ldots) = A_0 \times \mathrm{Str}(A_1, A_2, \ldots)$.

> "A stream with types $\mathbf{A} = A_0, A_1, A_2, \ldots$ is an element of A_0 together with a stream with types $A_1, A_2, \ldots$."

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Monoidal Categories

Cartesian monoidal categories encode these theories of processes that allow for copying and discarding.

- Plain functions are cartesian.
- Stochastic functions are not cartesian.
- Partial functions are not cartesian.

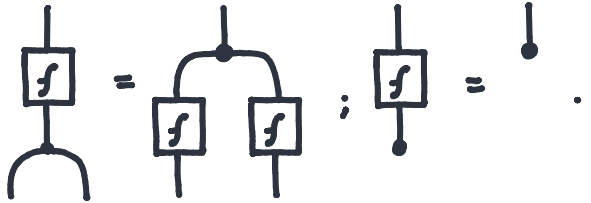

FIG 1. Copy and discard.

FIG. 2. Failure of stochastic copy; failure of partial discard.

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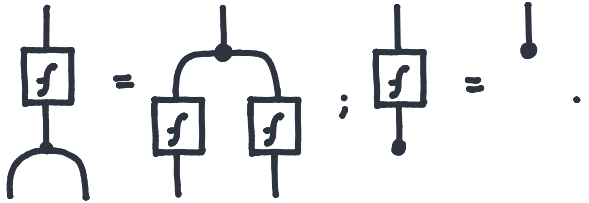

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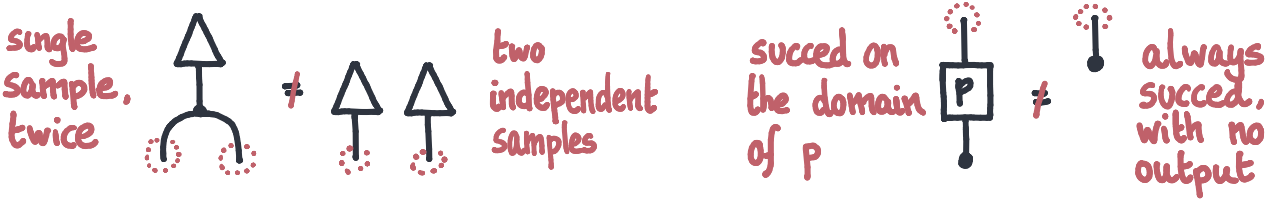

FIG. 2. Failure of stochastic copy; failure of partial discard.