Monoidal Context Theory

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Tallinn University of Technology

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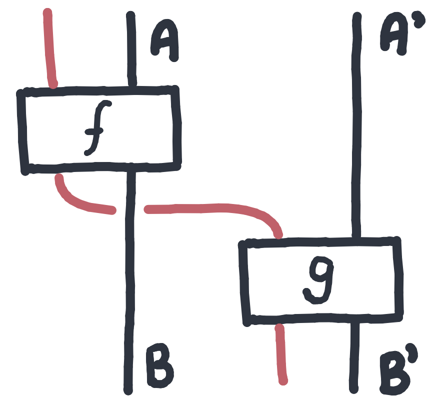
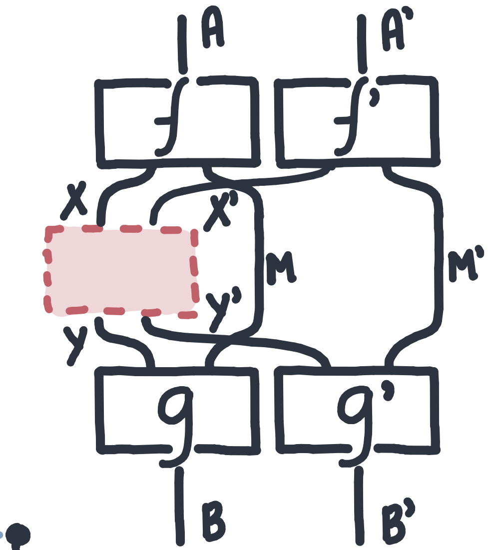
Outline

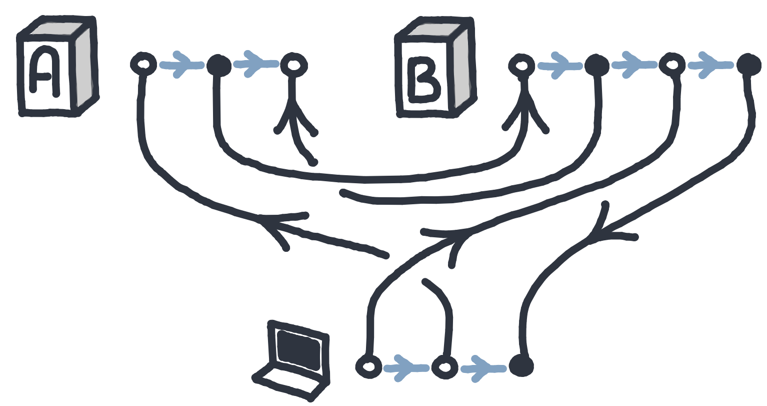
Process Theories

Monoidal Context

Message Theories

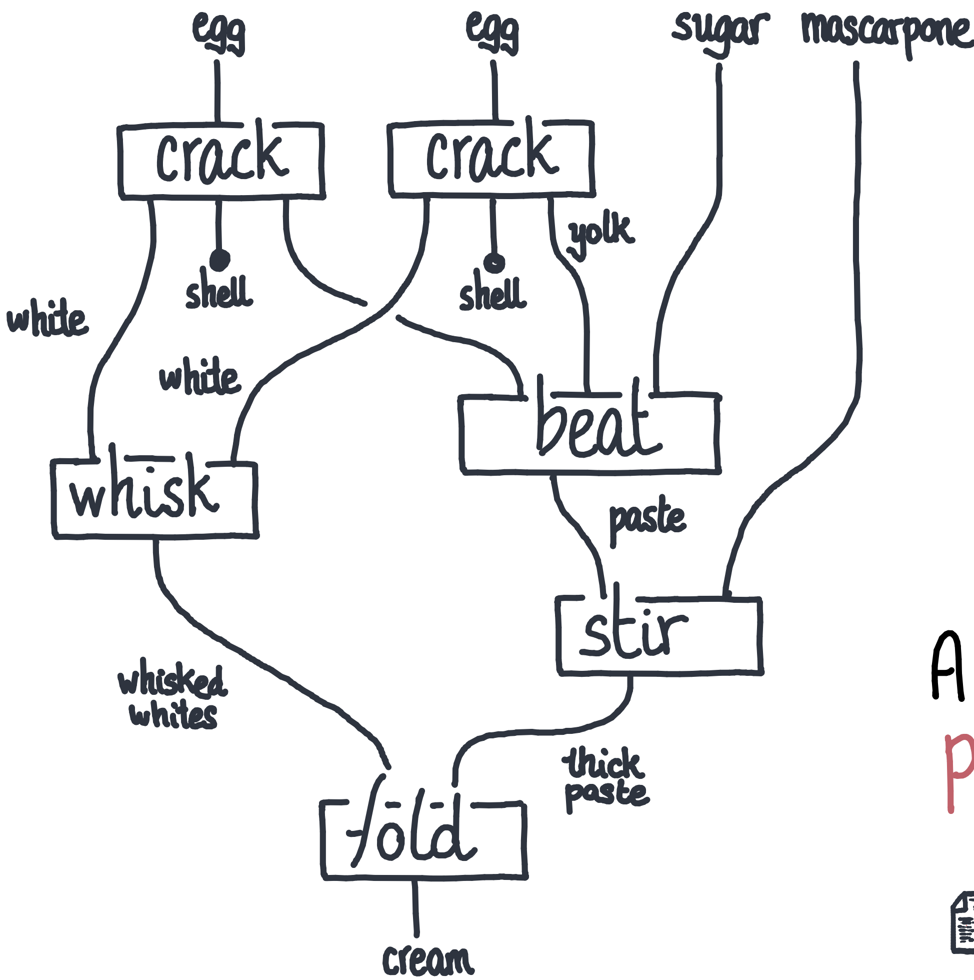






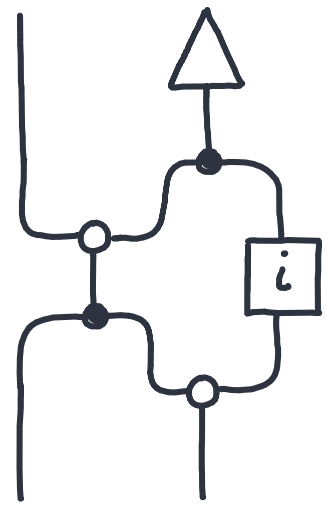
PART 0: Process Theories

EXAMPLES



Cooking recipe: crema di Mascarpone

Sobonciński

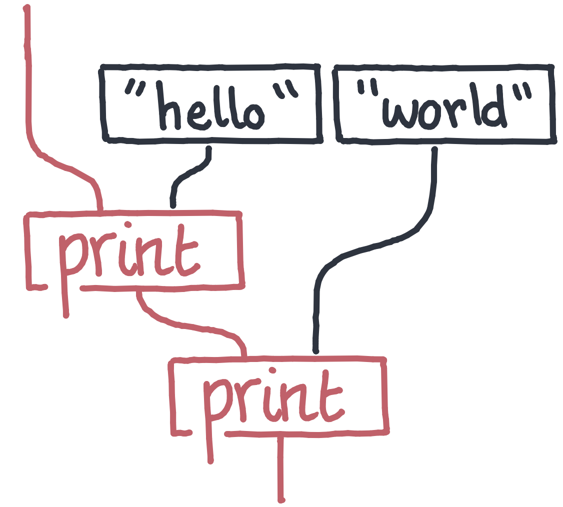


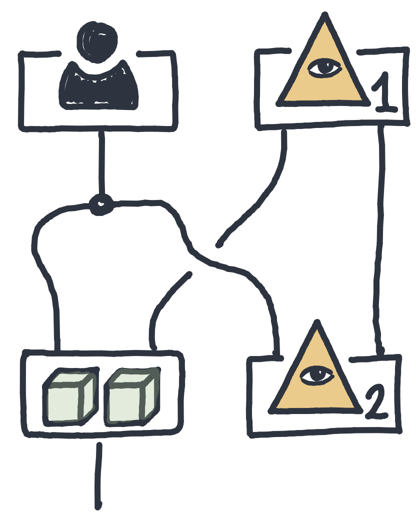
A cryptographic protocol: One-time Pad

Broadbent, Karvonen

An imperative program: "hello world"

Jeffrey



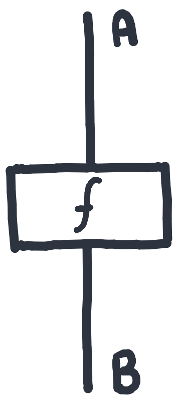


A decision problem: Newcomb's paradox

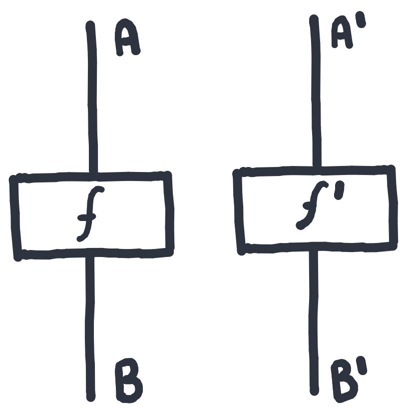
Di Lavore, Román

Monoidal Categories: Process Theories

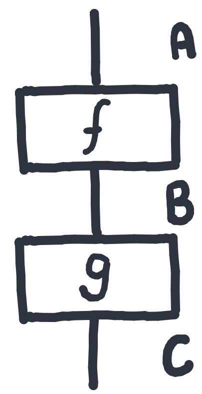
Monoidal categories are an algebra of parallel and sequential composition.
String diagrams are an internal language of monoidal categories.



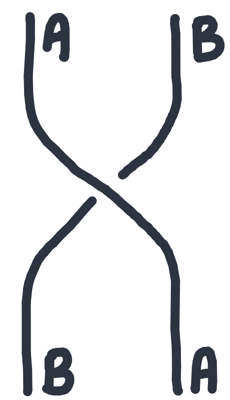
Process



Parallel composition



Sequential composition



Swap

Bènabou

Monoidal Categories: Process Theories

Monoidal categories are an algebra of parallel and sequential composition.
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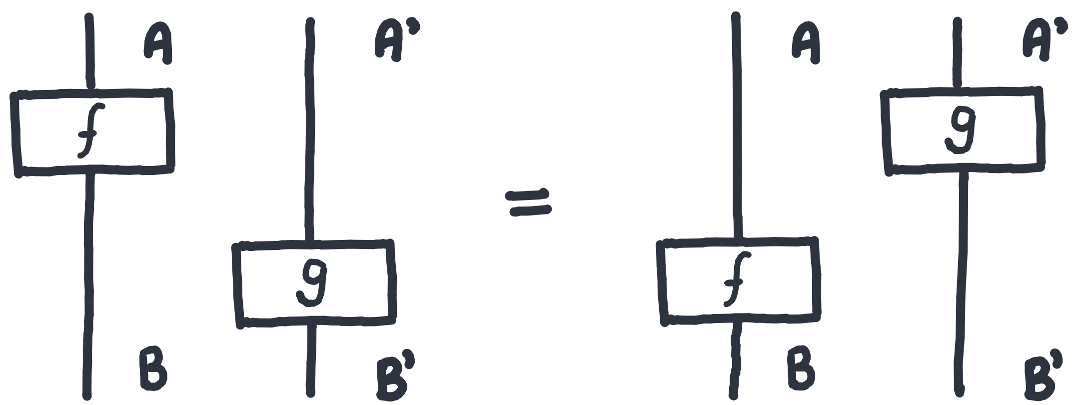
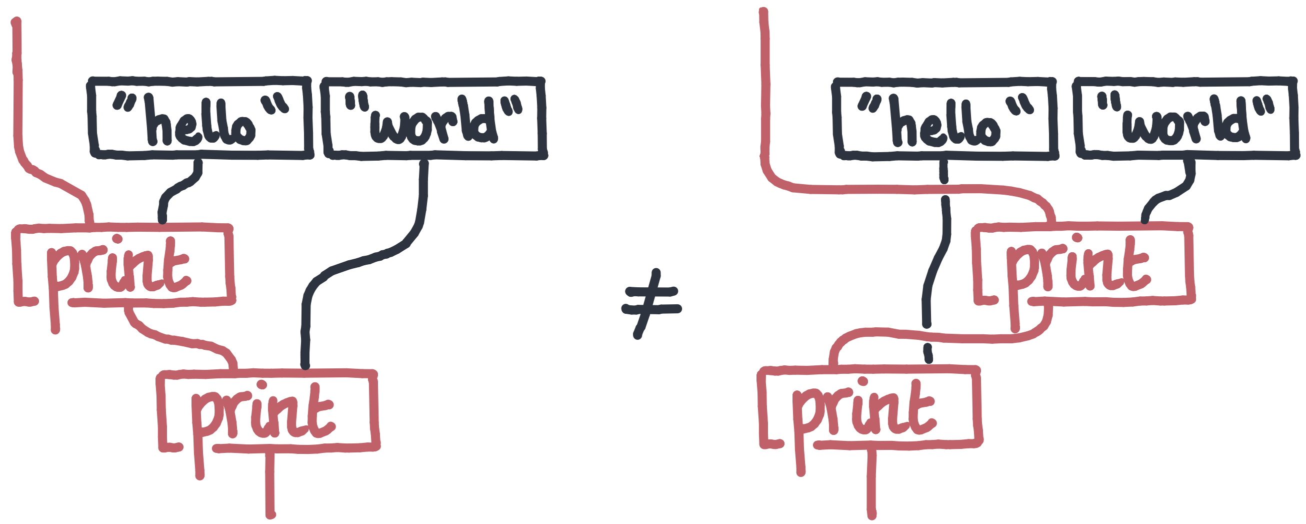


Interchange Law

Bènabou

Processes are Premonoidal

- Power, Robinson. Premonoidal Categories and Notions of Computation.
- Jeffrey. A Graphical View of Programs.
- Román. Promonads and String Diagrams for Effectful Categories. ACT'22.
- Staton, Møgelberg. Linear Usage of State.



THEOREM. String diagrams with runtime are the internal language of premonoidal categories.

PROCESSES ARE PREMONOIDAL

```
f() = do
    let a = "hello"
    let b = "world"
    print(a)
    print(b)
```

$\neq$

```
f() = do
    let b = "world"
    let a = "hello"
    print(b)
    print(a)
```

;

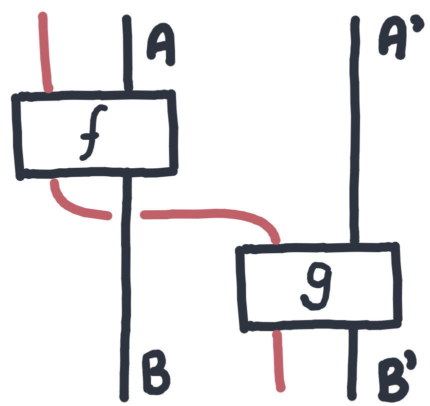
- Staton, Levy. Universal Properties of Impure Programming Languages.
- Jacobs, Hasuo. Freyd is Kleisli, for Arrows.
- Power, Robinson. Premonoidal Categories and Notions of Computation.
- Hughes. Generalising monads to arrows.

Do-Notation

$$\frac{}{a_0:A_0,\dots,a_n:A_n \gg_\tau () \vdash \mathsf{return}(a_0,\dots,a_n) : A_0\otimes\cdots\otimes A_n}\ \textsc{Return}_\tau$$

$$\frac{b_0:B_0,\dots,b_m:B_m,\ \Gamma \vdash t:\Delta}{a_0:A_0,\dots,a_n:A_n \gg_\tau \Gamma \vdash f(a_0,\dots,a_n)\to b_0,\dots,b_m\ ;\ t:\Delta}\ \textsc{Gen}_{f,\tau}$$

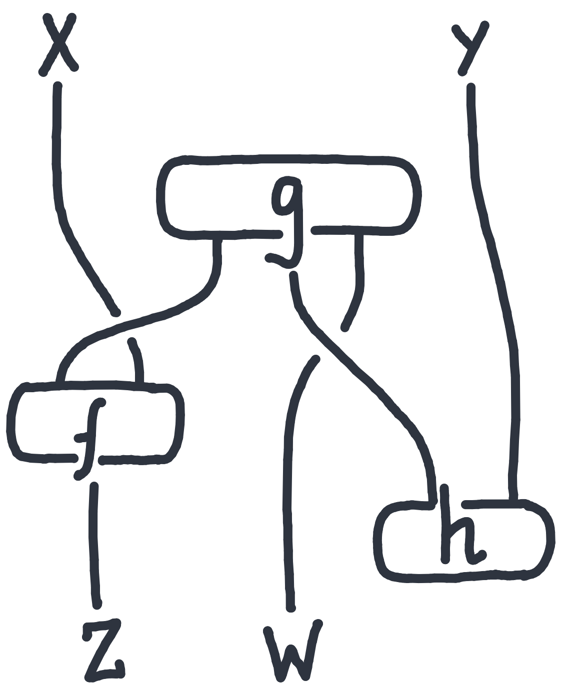
Linear do-notation is an internal language for sym. premonoidals.




```
(a,a') ⇒ do
    f(a) → b
    g(a') → b'
    return (b,b')
```

Do-Notation

Linear do-notation is an internal language for sym. premonoidals.
Adding interchange recovers sym. monoidals.




```
(x, y) ⇒ do
    g() → a,b,w
    f(a,x) → z
    h(b,y) → ()
    return (z,w) : Z⊗W
```

Do-Notation

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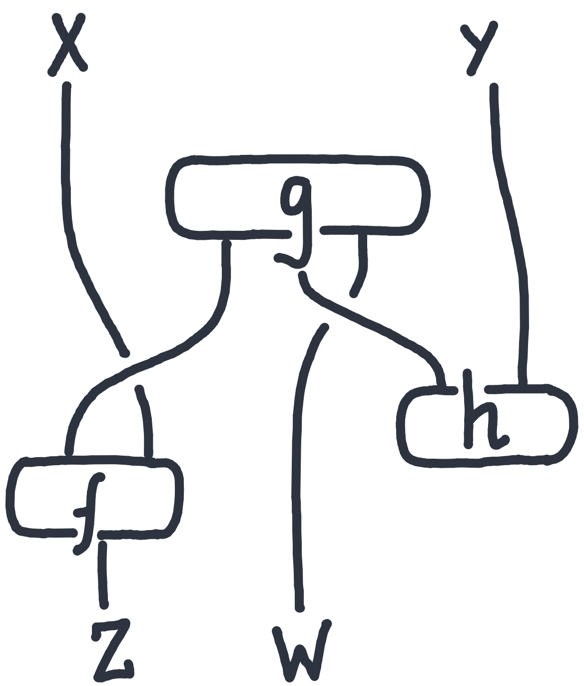


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Do-Notation

	Diagrams	Programs
Monoidal Category	String Diagrams	Linear Do-Notation with Interchange (Symm)
Premonoidal Category	String Diagrams with runtime	Linear Do-Notation (Symm)

PART 1: Optics

Optics

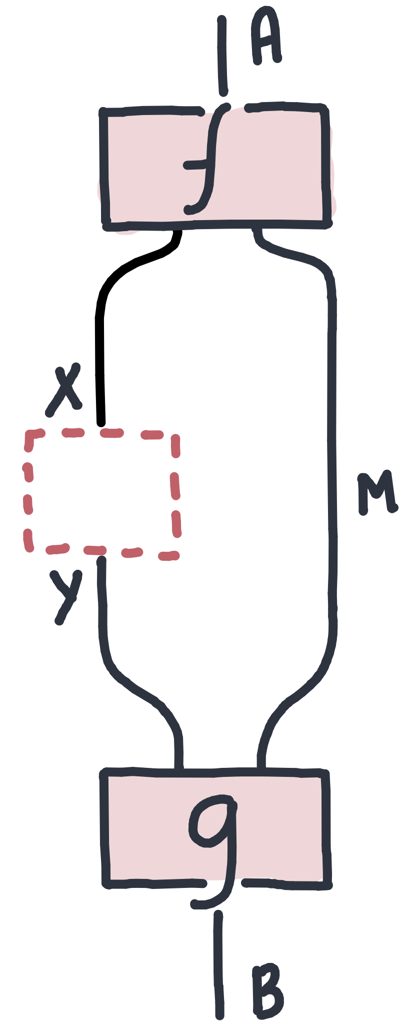
DEFINITION. Let $\mathbb{C}, \otimes, I$ symm. monoidal.

An optic from A to B with a hole from X to Y is a pair of morphisms

$$f : A \to X \otimes M, \qquad g : Y \otimes M \to B,$$

written as $\langle f \mid g \rangle$, and quotiented by dinaturality on M:

$$\langle f ; (id \otimes h) \mid g \rangle = \langle f \mid (id \otimes h) ; g \rangle.$$



Riley. Categories of optics.

OPTICS

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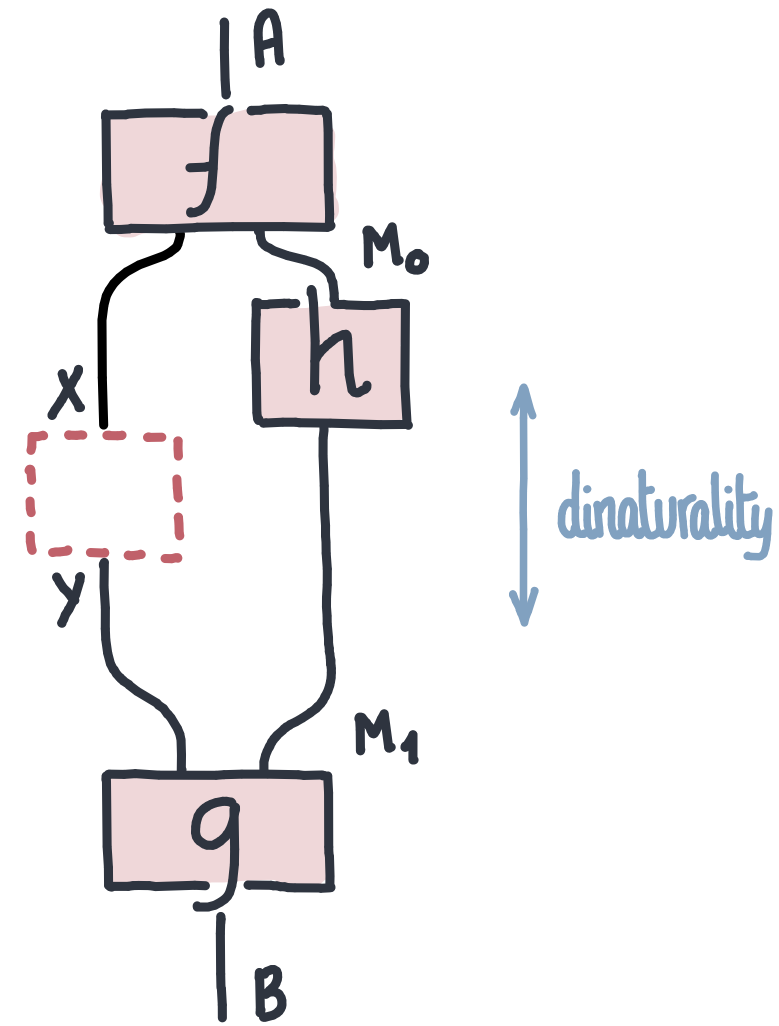

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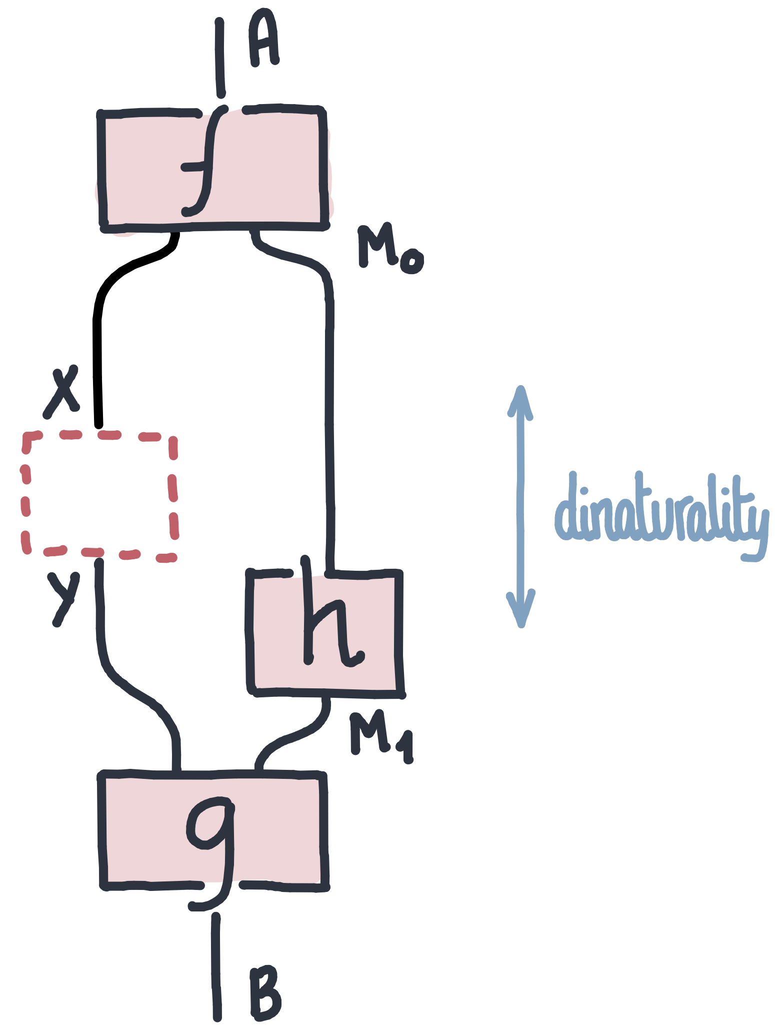
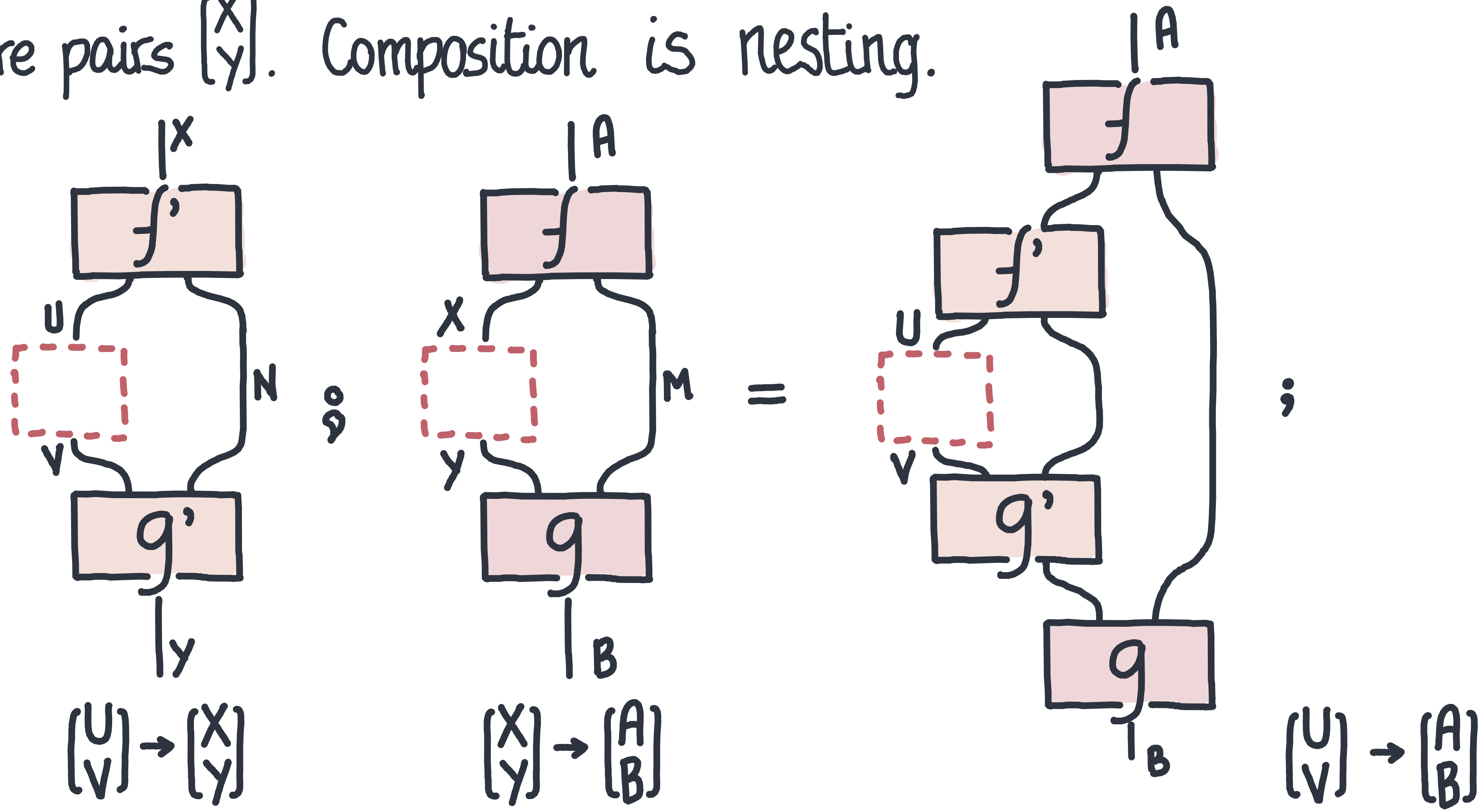


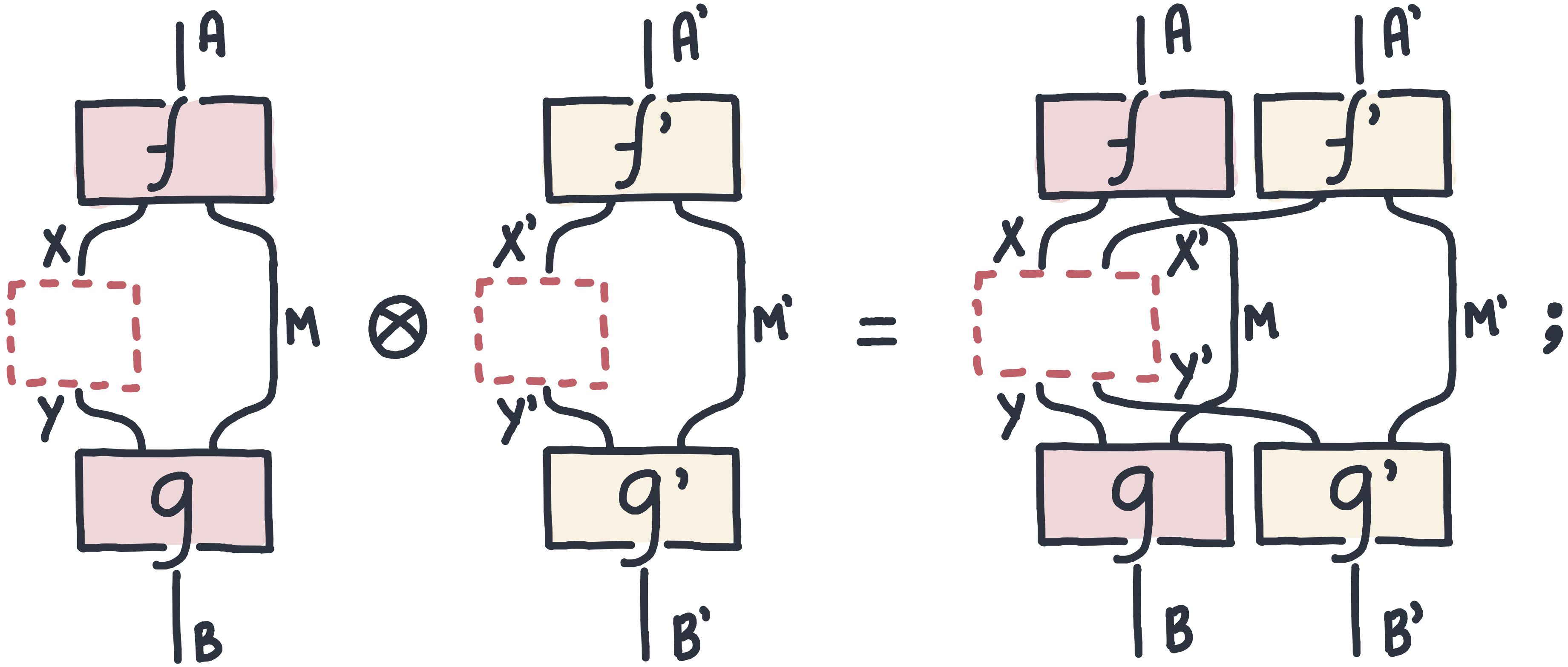
Optics form a Category

Objects are pairs $\begin{bmatrix} X \\ Y \end{bmatrix}$. Composition is nesting.



Optics form a Monoidal Category

Tensoring is $\begin{bmatrix}X\\Y\end{bmatrix}\otimes\begin{bmatrix}X'\\Y'\end{bmatrix}=\begin{bmatrix}X\otimes X'\\Y\otimes Y'\end{bmatrix}$, and



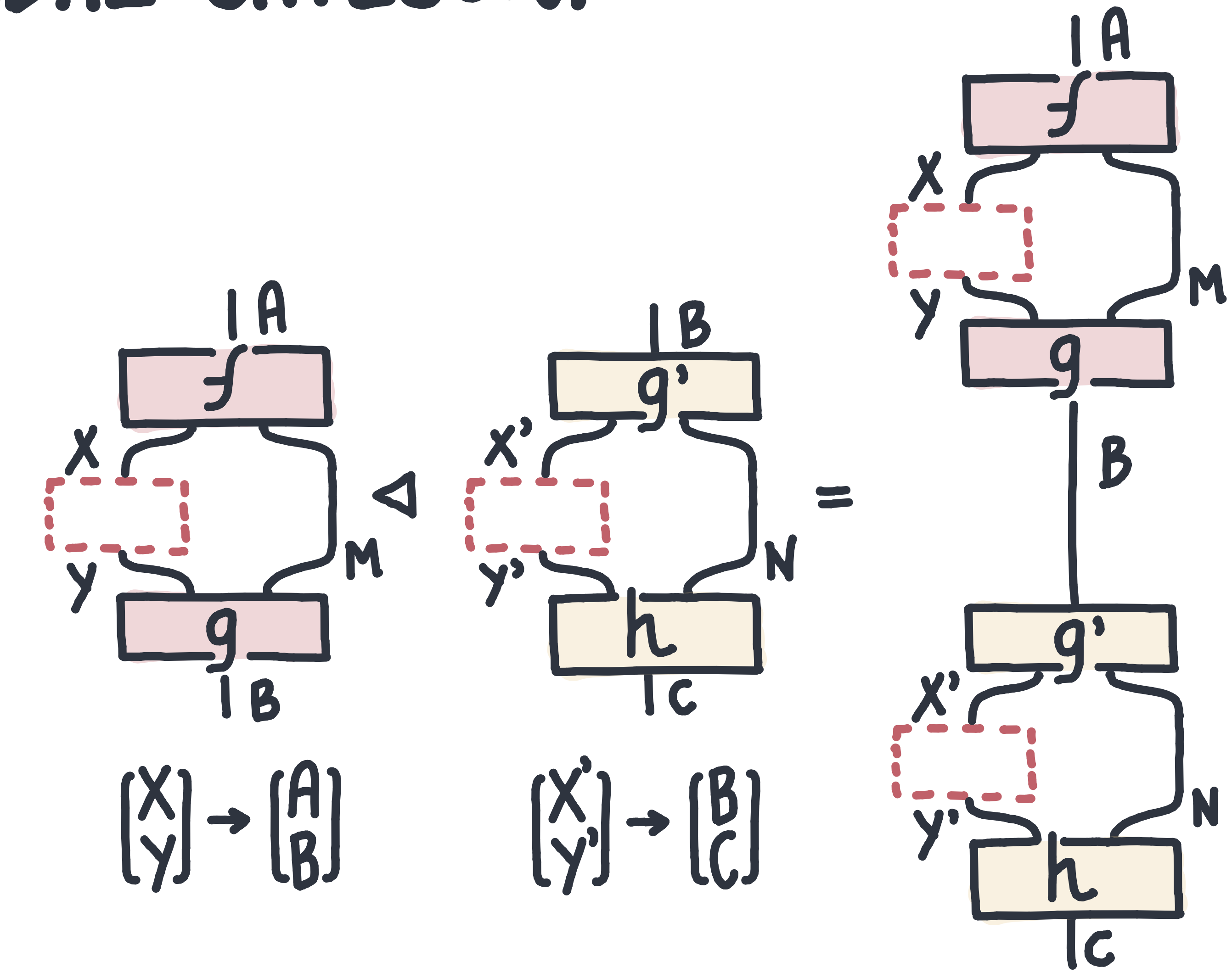
$\begin{bmatrix}X\\Y\end{bmatrix}\to\begin{bmatrix}A\\B\end{bmatrix}$ $\quad\begin{bmatrix}X'\\Y'\end{bmatrix}\to\begin{bmatrix}A'\\B'\end{bmatrix}$ $\quad\begin{bmatrix}X\otimes X'\\Y\otimes Y'\end{bmatrix}\to\begin{bmatrix}A\otimes A'\\B\otimes B'\end{bmatrix}$

Optics form a Duoidal Category?

Sequencing is not an operation:

$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix}$ is not an object, even when $\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$ is defined.

This is not monoidal, but it is still promonoidal.



$\begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix}$ $\begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} B \\ C \end{bmatrix}$ $\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ C \end{bmatrix}$

Optics form a Duoidal Category?

Sequencing is not an operation, it defines a hom-set to an object that does not really exist.

$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix}$ is not an object,

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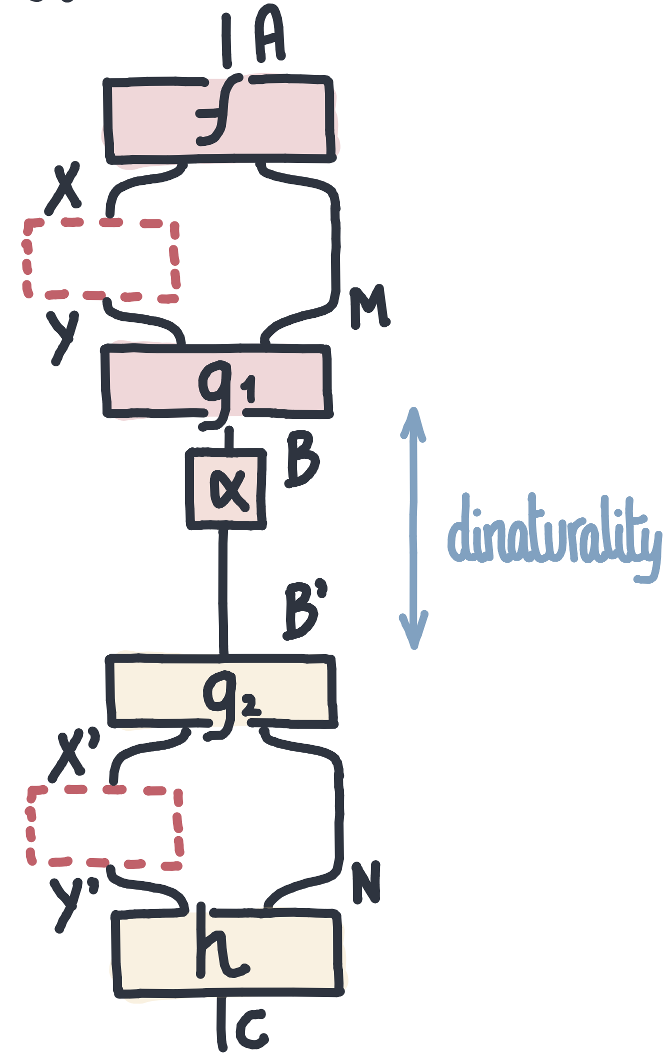

$$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$$

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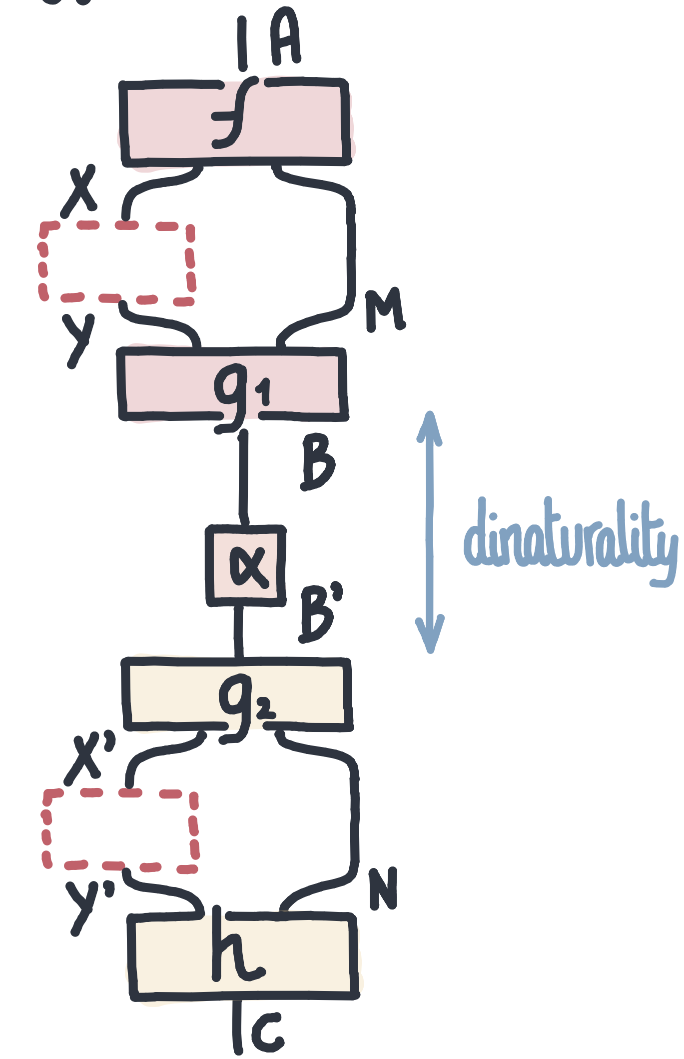

$$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ C \end{bmatrix}$$

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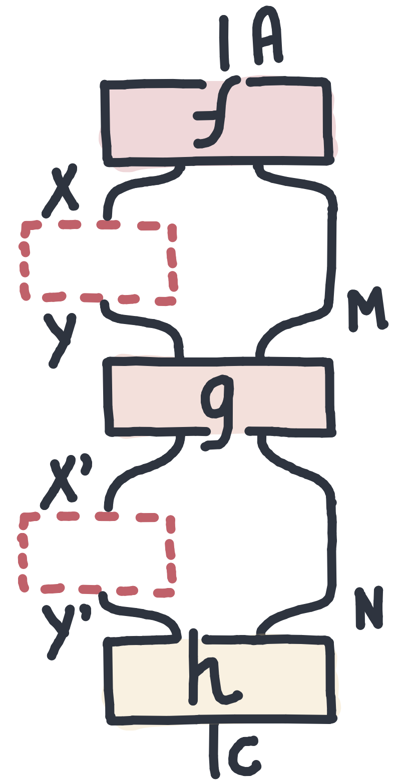


$$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ C \end{bmatrix}$$

Promonoidal Categories

Monoidal Category

DEFINITION. A monoidal category is a category $\mathbb{C}$ together with functors

$$(\otimes) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \qquad I : \mathbb{1} \to \mathbb{C},$$

and natural isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$
$$\lambda_A : I \otimes A \to A,$$
$$\rho_A : A \otimes I \to A,$$

satisfying the pentagon and triangle equations.

By nesting, $X \otimes (Y \otimes Z)$, we mean functor composition,

$$X \otimes (Y \otimes Z) :=$$
$$X \otimes M \text{ where } M = Y \otimes Z.$$

Promonoidal Category

DEFINITION. A promonoidal category is a category $\mathbb{C}$ together with profunctors

$$\mathbb{C}(\bullet \otimes \bullet ; \bullet) \colon \mathbb{C}^{op} \times \mathbb{C}^{op} \times \mathbb{C} \to \mathrm{SET}, \qquad \mathbb{C}(I ; \bullet) \colon \mathbb{C}^{op} \to \mathrm{SET},$$

and natural bijections,

$$\alpha_{A,B,C} \colon \mathbb{C}(X \otimes (Y \otimes Z) ; \bullet) \to \mathbb{C}((X \otimes Y) \otimes Z ; \bullet),$$
$$\lambda_A \colon \mathbb{C}(I \otimes X ; \bullet) \to \mathbb{C}(X ; \bullet),$$
$$\rho_A \colon \mathbb{C}(X \otimes I ; \bullet) \to \mathbb{C}(X ; \bullet),$$

satisfying the pentagon and triangle equations.

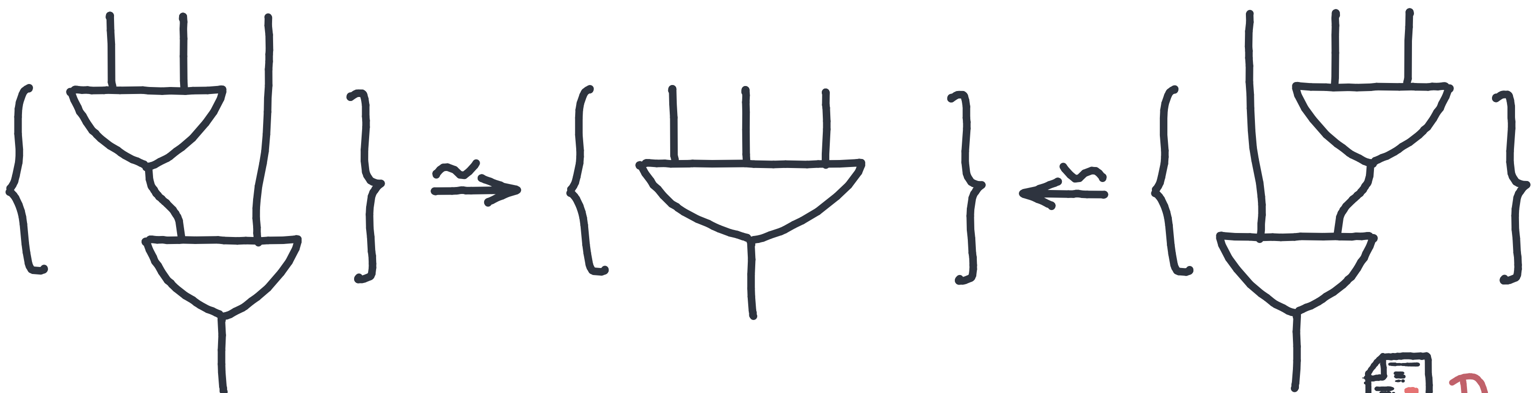
By nesting, $\mathbb{C}(X \otimes (Y \otimes Z) ; \bullet)$, we mean profunctor composition,

$$\mathbb{C}(X \otimes (Y \otimes Z) ; \bullet) := \int^{M} \mathbb{C}(X \otimes M ; \bullet) \times \mathbb{C}(Y \otimes Z ; M).$$

MALLEABLE MULTICATEGORIES

DEFINITION. A malleable multicategory is a multicategory where dinatural composition is invertible,

$$(\fatsemi) : \left(\int^{y \in \mathbb{M}} \mathbb{M}(\Gamma; y) \times \mathbb{M}(\Delta, y, \Delta'; Z)\right) \to \mathbb{M}(\Delta, \Gamma, \Delta'; Z).$$



Day, Panchadcharam, Street

THEOREM. The full subcategory of malleable multicategories is equivalent to the category of promonoidal categories.

PART 2 : Context Theory

Contour is Adjoint to Splice

What is a canonical algebra of context on top of a category?

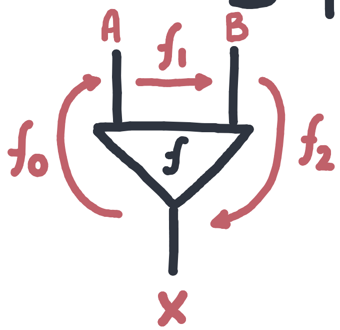
- Each promonoidal gives a free category, contour.
- Each category gives a cofree promonoidal, context.

$$
\begin{tikzcd}
\text{Category} \arrow[r, bend left, "\text{Context}"] \arrow[r, phantom, "\top"] & \text{Promonoidal Category} \arrow[l, bend left, "\text{Contour}"]
\end{tikzcd}
$$

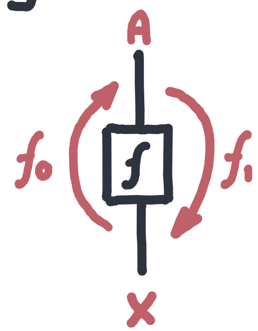
CONTOUR

Melliès, Zeilberger.

Contouring promonoidal categories generates a category.



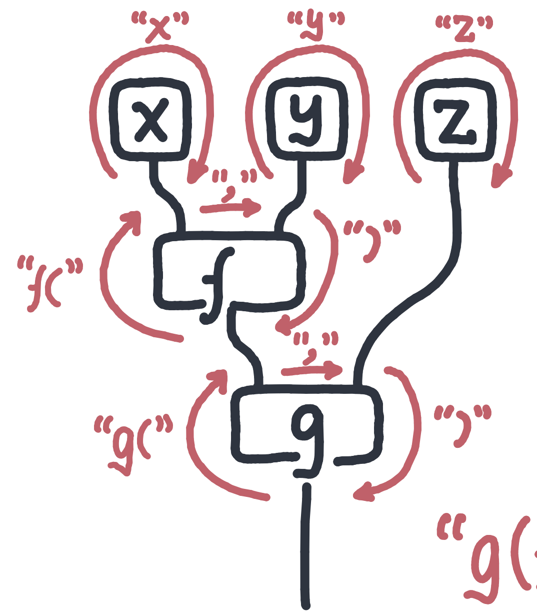
$$f_0 : X^L \to A^L$$
$$f_1 : A^R \to B^L$$
$$f_2 : B^R \to X^R$$



$$f_0 : X^L \to A^L$$
$$f_1 : A^R \to X^R$$

$$f_0 : X^L \to X^R.$$

The category provides
a simple parsing algebra to
any promonoidal,
or any multicategory.



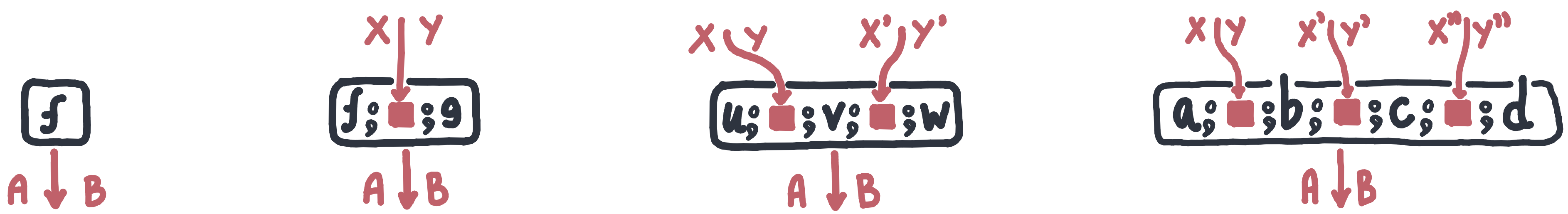
"g(f(x,y),z)"

Context for Categories

Consider 'expressions with holes' in a category, like the following

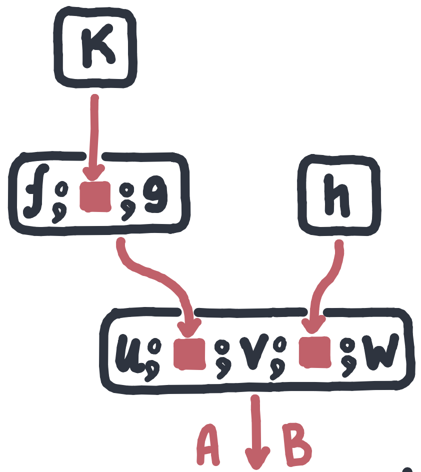
$$u ; \blacksquare ; v ; \blacksquare ; w, \qquad f ; \blacksquare ; g, \qquad f, \qquad a ; \blacksquare ; b ; \blacksquare ; c ; \blacksquare ; d.$$

These contexts form a promonoidal category.



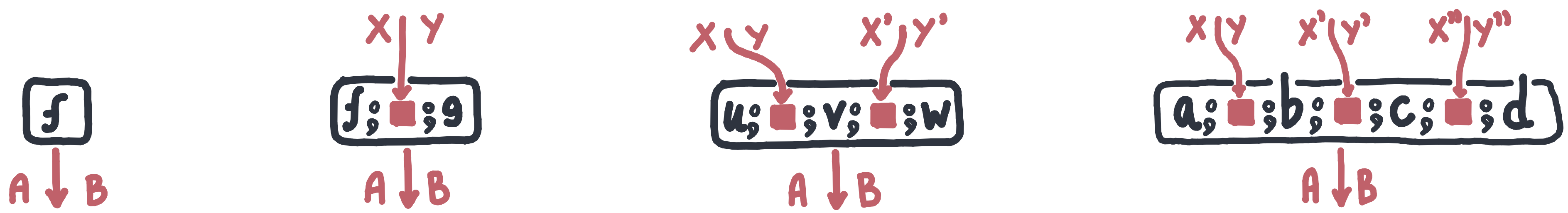
$$\begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \triangleleft \begin{bmatrix}X''\\Y''\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$$

Context for Categories



$= u ; f ; k ; g ; v ; h ; w$

These contexts form a promonoidal category.



$\begin{bmatrix}A\\B\end{bmatrix}$ $\quad$ $\begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad$ $\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$ $\quad$ $\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \triangleleft \begin{bmatrix}X''\\Y''\end{bmatrix} \to \begin{bmatrix}A\\C\end{bmatrix}$

Contour is Adjoint to Context

What is a canonical algebra of context on top of a category?

- Each category gives a cofree promonoidal, context.
- Each promonoidal gives a free category, contour.

$$\begin{tikzcd}
\text{Category} \arrow[r, bend left, "\text{Context}"] \arrow[r, phantom, "\perp"] & \text{Promonoidal Category} \arrow[l, bend left, "\text{Contour}"]
\end{tikzcd}$$

Part 3: Context for Monoidal Categories

Monoidal Context-Contour

What is a canonical algebra of decomposition on top of a monoidal category?

- Each monoidal category gives a cofree produoidal, monoidal context.
- Each produoidal gives a free monoidal category, monoidal contour.

$$
\begin{tikzcd}
\text{Monoidal Category} \arrow[r, bend left, "\text{Monoidal Context}"] \arrow[r, phantom, "\top"] & \text{Produoidal Category} \arrow[l, bend left, "\text{Monoidal Contour}"]
\end{tikzcd}
$$

PRODUOIDAL CATEGORIES

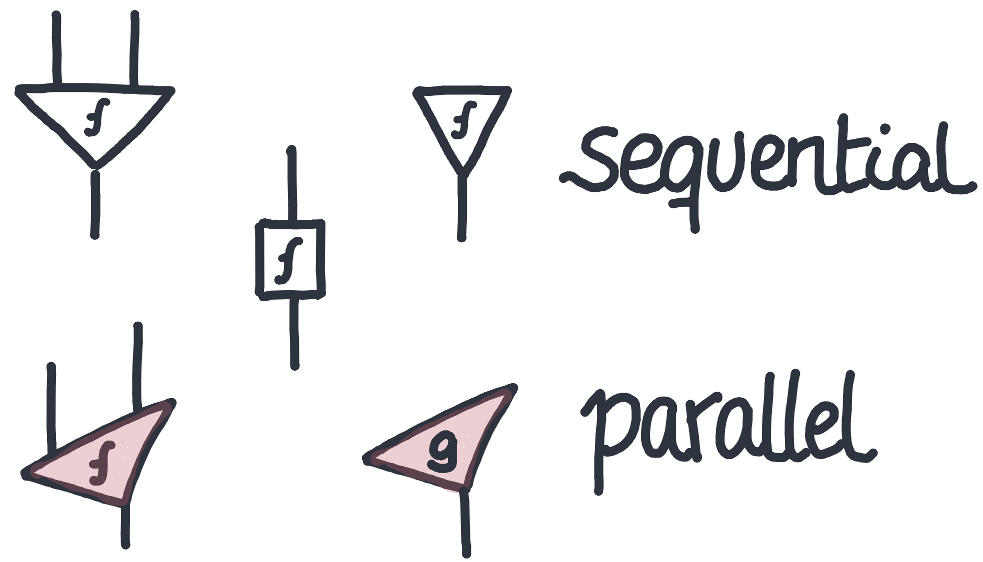
DEFINITION. A produoidal is a pair of promonoidals

$\mathbb{V}(\bullet \triangleleft \bullet ; \bullet) : \mathbb{V}^{op} \times \mathbb{V} \times \mathbb{V} \to \mathrm{SET}$, $\mathbb{V}(\bullet ; N) : \mathbb{V}^{op} \to \mathrm{SET}$, "sequential",

$\mathbb{V}(\bullet ; \bullet \otimes \bullet) : \mathbb{V}^{op} \times \mathbb{V} \times \mathbb{V} \to \mathrm{SET}$, $\mathbb{V}(\bullet ; I) : \mathbb{V}^{op} \to \mathrm{SET}$, "parallel".

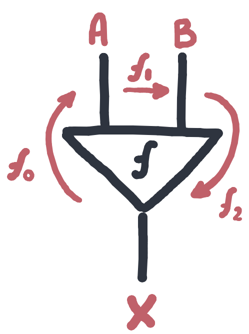
One laxly distributes over the other,

$$\psi_2 : (A \triangleleft B) \otimes (C \triangleleft D) \longrightarrow (A \otimes C) \triangleleft (B \otimes D),$$
$$\psi_0 : I \longrightarrow N$$
$$\varphi_2 : N \longrightarrow N \triangleleft N$$
$$\varphi_0 : I \longrightarrow I \otimes I$$



Monoidal Contour

Contouring produoidal categories generates a monoidal category.



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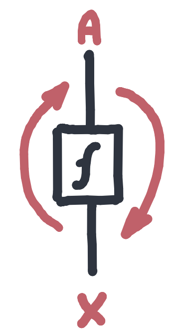
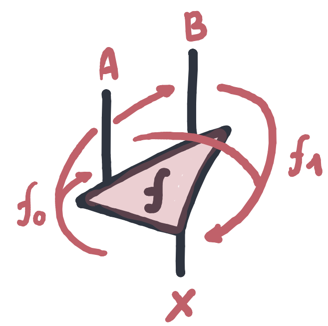
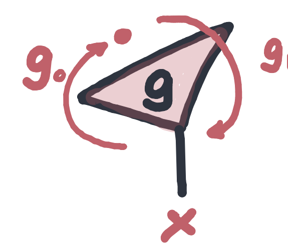


$f_0 : X^L \to A^L$
$f_1 : A^R \to X^R$

$f_0 : X^L \to X^R$



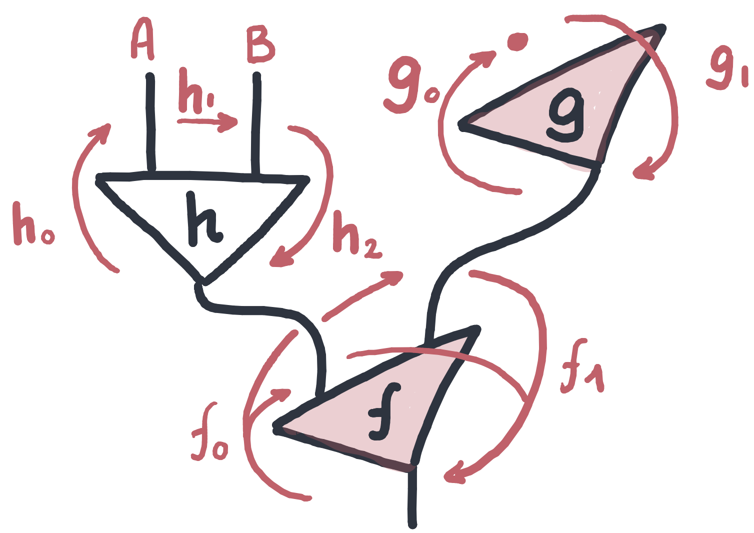
$f_0 : X^L \to A^L \otimes B^L$
$f_1 : A^R \otimes B^R \to X^R$

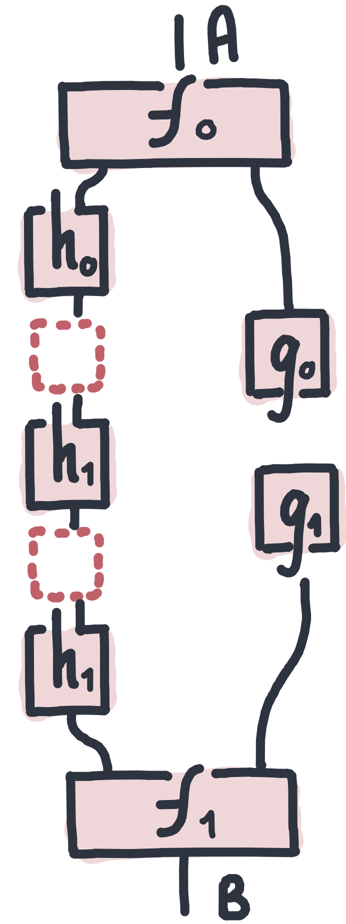


$g_0 : X^L \to I$
$g_1 : I \to X^R$

Monoidal Contour

Contouring produoidal categories generates a monoidal category. Example.



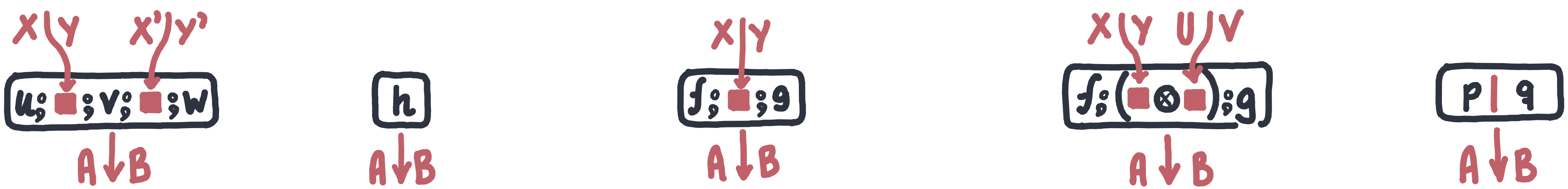


Monoidal Context

Consider 'expressions with holes' in a monoidal category, like the following

$$u ⨾ ■ ⨾ v ⨾ ■ ⨾ w, \qquad k, \qquad f ⨾ (■ \otimes ■) ⨾ g, \qquad p \mid q.$$

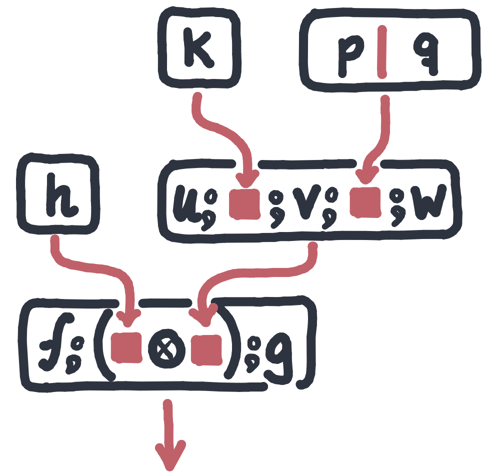
These contexts form a produoidal category.



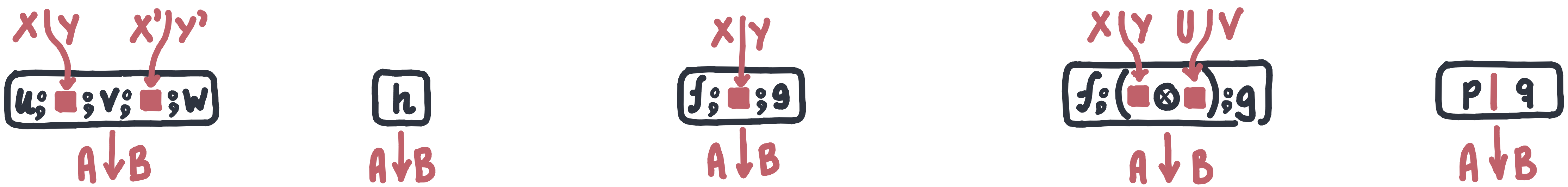
$$\begin{bmatrix} X \\ Y \end{bmatrix} \triangleleft \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix} \qquad N \to \begin{bmatrix} A \\ B \end{bmatrix} \qquad \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix} \qquad \begin{bmatrix} X \\ Y \end{bmatrix} \otimes \begin{bmatrix} X' \\ Y' \end{bmatrix} \to \begin{bmatrix} A \\ B \end{bmatrix} \qquad I \to \begin{bmatrix} A \\ B \end{bmatrix}$$

Monoidal Context

$$f ; (h \otimes (u ; k ; v ; p ; q ; w)) ; g \quad = $$



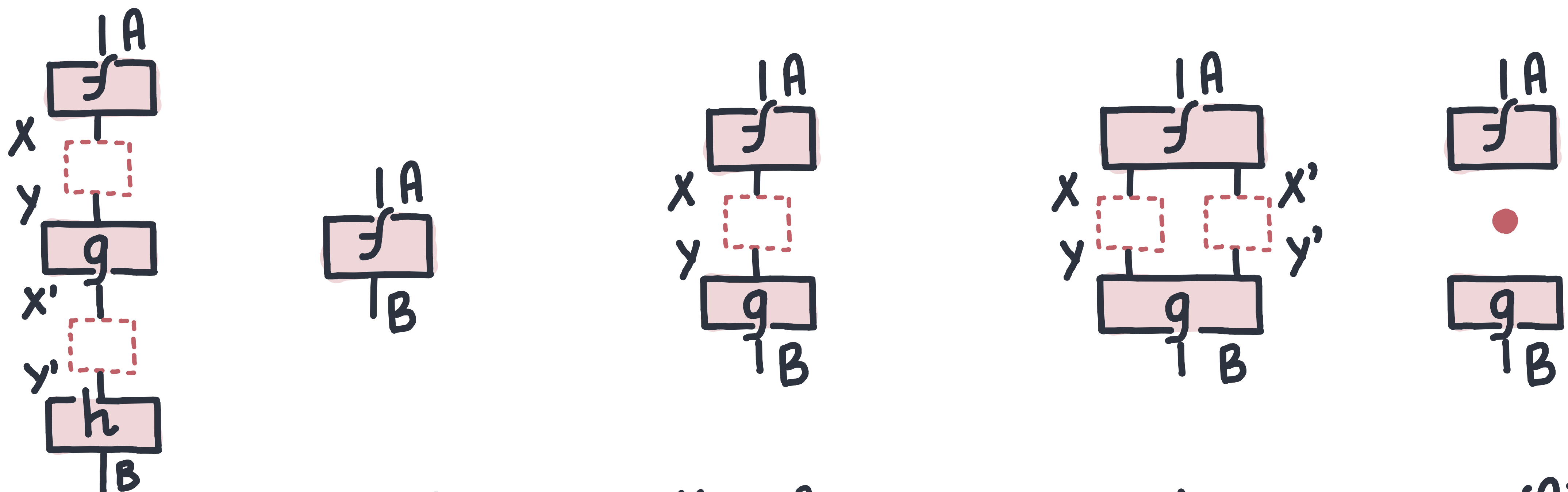
These contexts form a produoidal category.



$$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad N \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad I \to \begin{bmatrix}A\\B\end{bmatrix}$$

Monoidal Context

Theorem. Spliced monoidal arrows are the cofree produoidal on a monoidal.

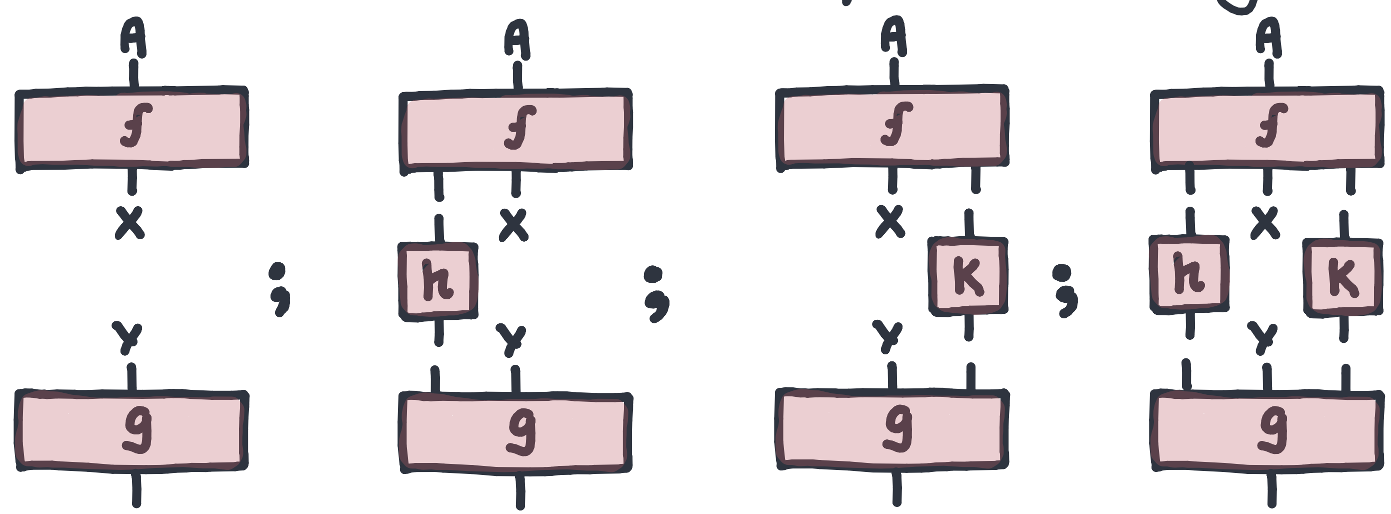


$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad N \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad I \to \begin{bmatrix}A\\B\end{bmatrix}$

MISSING

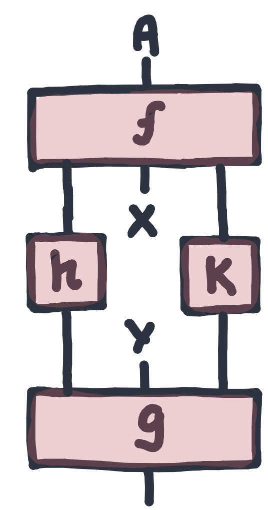
Spliced monoidal arrows have some issues:

- They separate sequential and parallel units unnecessarily.
- Produoidals introduce a lot of bureaucracy on units.



$\begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$; $N \otimes \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$; $\begin{bmatrix}X\\Y\end{bmatrix} \otimes N \to \begin{bmatrix}A\\B\end{bmatrix}$; $N \otimes \begin{bmatrix}X\\Y\end{bmatrix} \otimes N \to \begin{bmatrix}A\\B\end{bmatrix}$

but we just want



$\mathbb{MC}\left(\begin{smallmatrix}A\\B\end{smallmatrix};\begin{smallmatrix}X\\Y\end{smallmatrix}\right)$.

NEXT

$$
\begin{tikzcd}
\text{Monoidal Category} \arrow[r, bend left, "\text{Mon. Splice}"] \arrow[r, phantom, "\top"] \arrow[dd] & \text{Produoidal Category} \arrow[l, bend left, "\text{Mon. Contour}"] \arrow[r, bend left, "\text{Normalization}"] \arrow[r, phantom, "\perp"] \arrow[dd] & \text{Normal Produoidal Category} \arrow[l, bend left, "\text{Forget}"] & \\
 & & \text{Duoidal Category} \arrow[ul, dashed] & \text{Normal Duoidal Category} \arrow[l, "\text{Forget}"] \arrow[ul, dashed] \\
\text{Category} \arrow[r, bend left, "\text{Splice}"] \arrow[r, phantom, "\top"] & \text{Promonoidal Category} \arrow[l, bend left, "\text{Contour}"] & &
\end{tikzcd}
$$

Part 4 : Duoidals

Duoidal Categories

Duoidal categories have two tensors and one distributes over the other. We interpret

- $(\triangleleft, N)$ as sequential tensor, "X, and then Y";
- $(\otimes, I)$ as parallel tensor, "X and Y at the same time".

When the unit distributor is an isomorphism, $I \xrightarrow{\sim} N$, it is normal. A normal, $\otimes$-symmetric duoidal, is a physical duoidal.

Aguiar & Mahajan '10

Normalizing Duoidals

THEOREM (Garner, López Franco). Let $(\mathbb{V}, \otimes, I, \triangleleft, N)$ a duoidal with reflexive coequalizers, preserved by $(\otimes)$. Then, $(\mathrm{Bimod}_N^{\otimes}, \otimes_N, N, \triangleleft, N)$ is a normal duoidal.

N is the unit of the category of $N^{\otimes}$-bimodules, which is monoidal,

$$N \otimes (A \triangleleft B) \otimes N \to (N \triangleleft N) \otimes (A \triangleleft B) \otimes (N \triangleleft N) \to (N \otimes A \otimes N) \triangleleft (N \otimes B \otimes N) \to A \triangleleft B,$$

but we require reflexive coequalizers to define the new tensor $(\otimes_N)$, and these coequalizers need to be preserved by $(\otimes)$,

$$A \otimes N \otimes B \rightrightarrows A \otimes B \twoheadrightarrow A \otimes_N B.$$

Normalizing Produoidals

THEOREM. We can ALWAYS normalize a produoidal category.

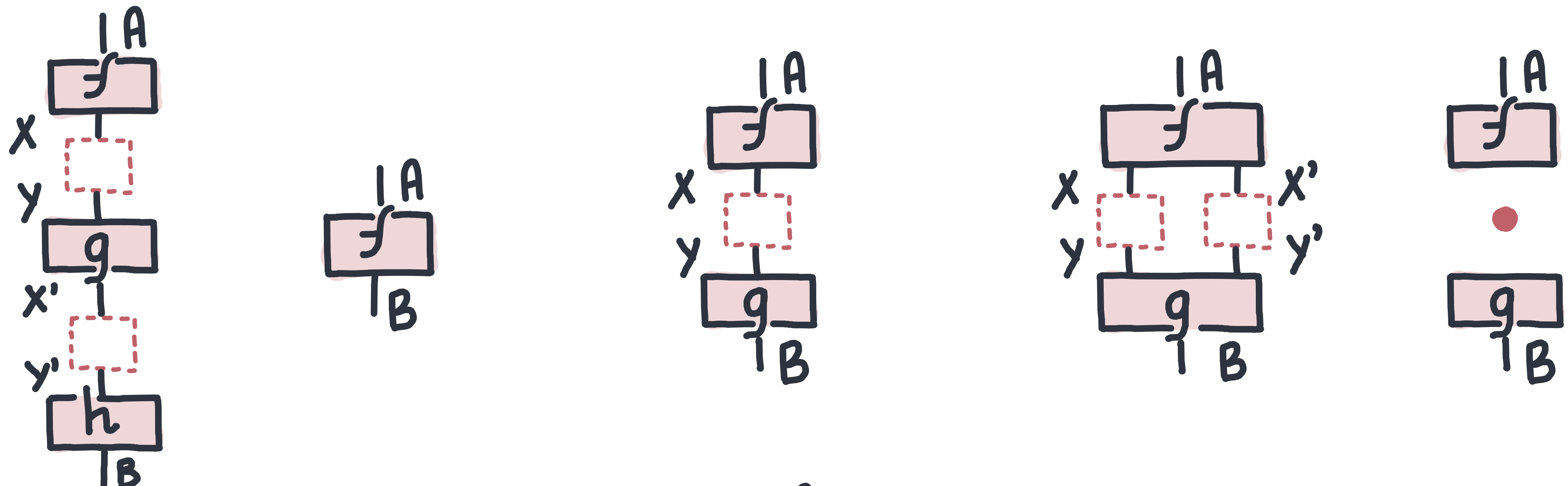
$$\mathcal{N}or\,\mathbb{V}(X;Y) = \mathbb{V}(X;N \otimes Y \otimes N),$$

Moreover, $\mathcal{N}or$: Produo $\to$ Produo is an idempotent monad, constructing a free normalization.

It is not that only some duoidals are normalizable. Every duoidal is normalizable, but the result may be a produoidal.

Monoidal Context

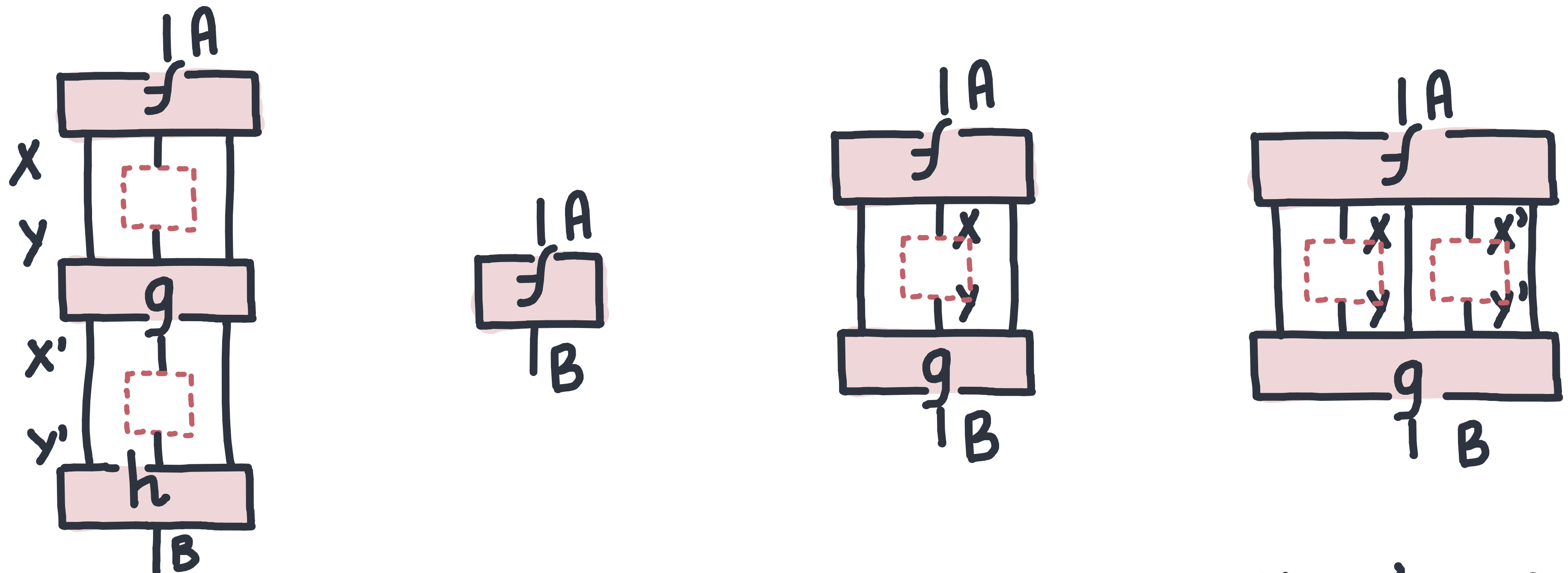
THEOREM. Monoidal context is the cofree produoidal on a monoidal.



$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad N \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad \begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$ $\quad I \to \begin{bmatrix}A\\B\end{bmatrix}$

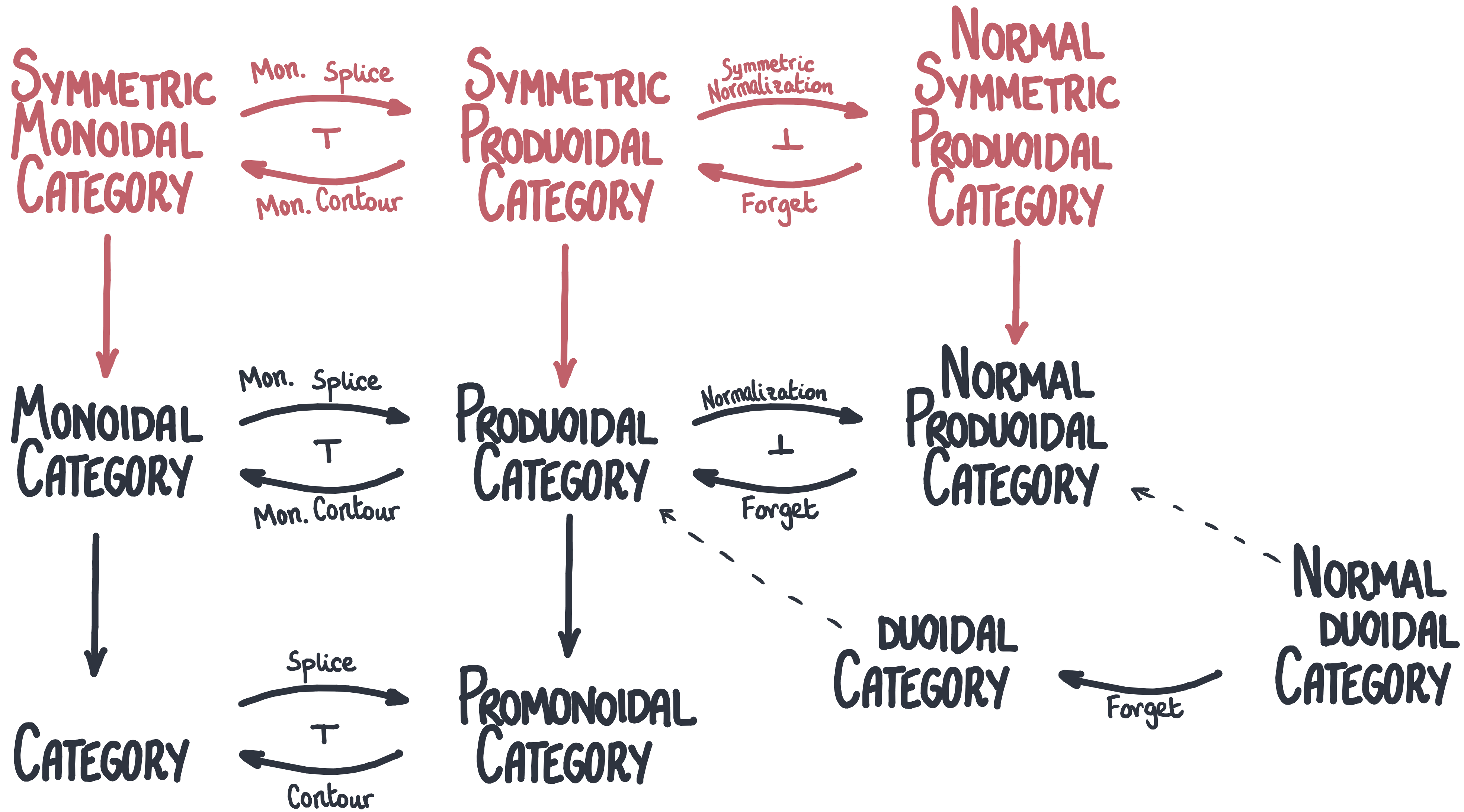
Normalized Monoidal Context

THEOREM. Monoidal optics are the free normalization of monoidal context.



$$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix} \qquad \begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$$

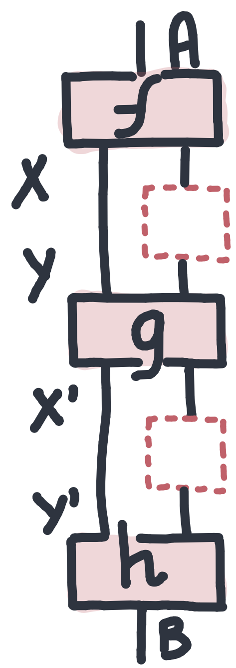
NEXT



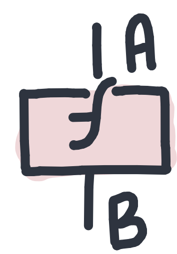
Symmetric Monoidal Category
Mon. Splice
⊤
Mon. Contour
Symmetric Produoidal Category
Symmetric Normalization
⊥
Forget
Normal Symmetric Produoidal Category
Monoidal Category
Mon. Splice
⊤
Mon. Contour
Produoidal Category
Normalization
⊥
Forget
Normal Produoidal Category
Category
Splice
⊤
Contour
Promonoidal Category
Duoidal Category
Forget
Normal Duoidal Category

Normalized Symmetric Monoidal Context

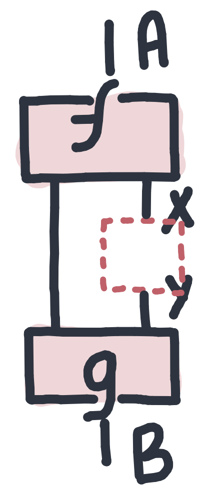
Theorem. Monoidal optics are the free sym normalization of sym monoidal context.



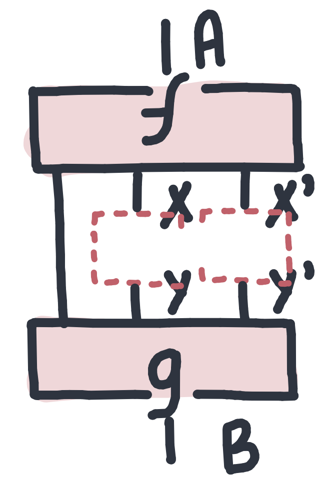
$\begin{bmatrix}X\\Y\end{bmatrix} \triangleleft \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$



$\begin{bmatrix}A\\B\end{bmatrix}$

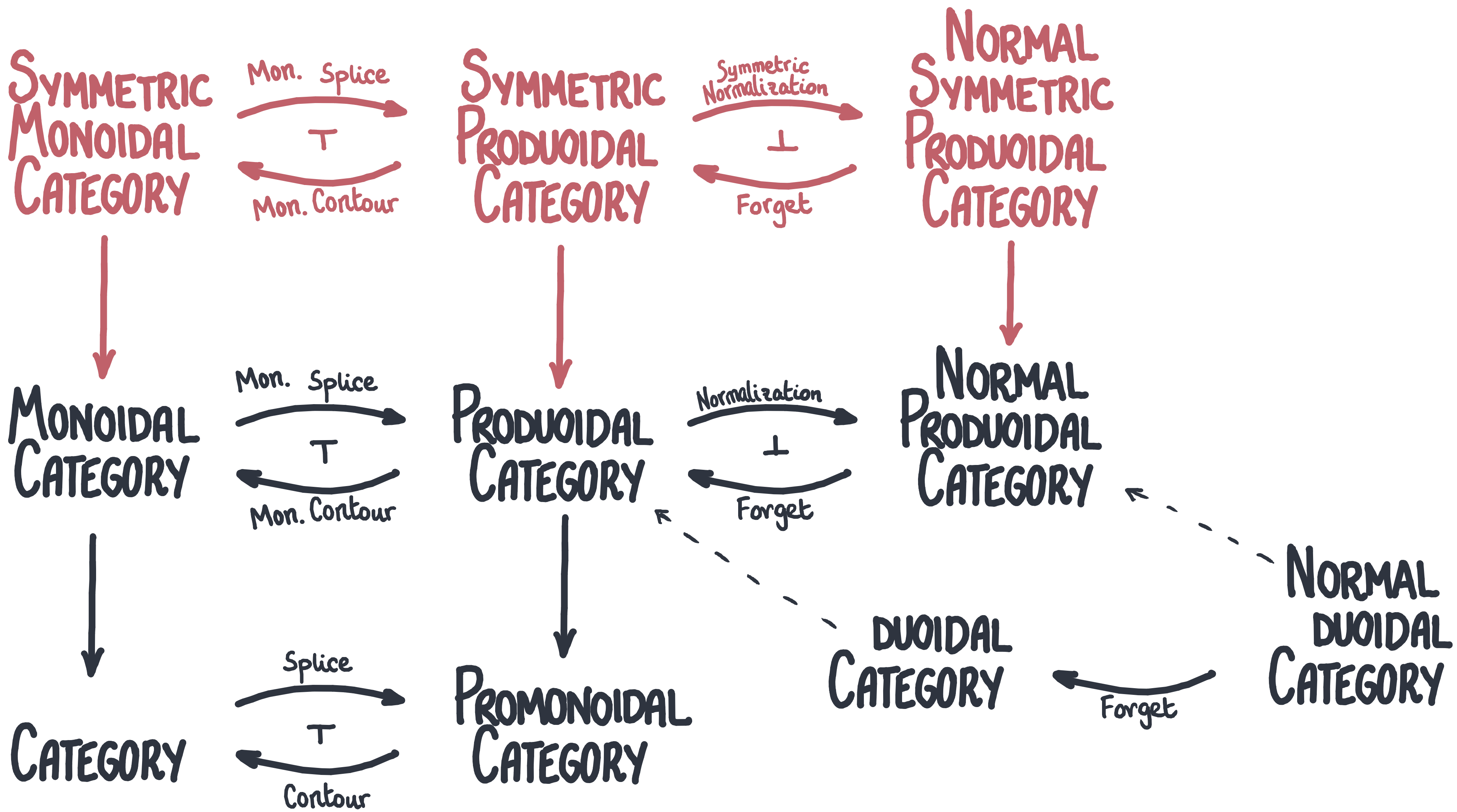


$\begin{bmatrix}X\\Y\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$



$\begin{bmatrix}X\\Y\end{bmatrix} \otimes \begin{bmatrix}X'\\Y'\end{bmatrix} \to \begin{bmatrix}A\\B\end{bmatrix}$

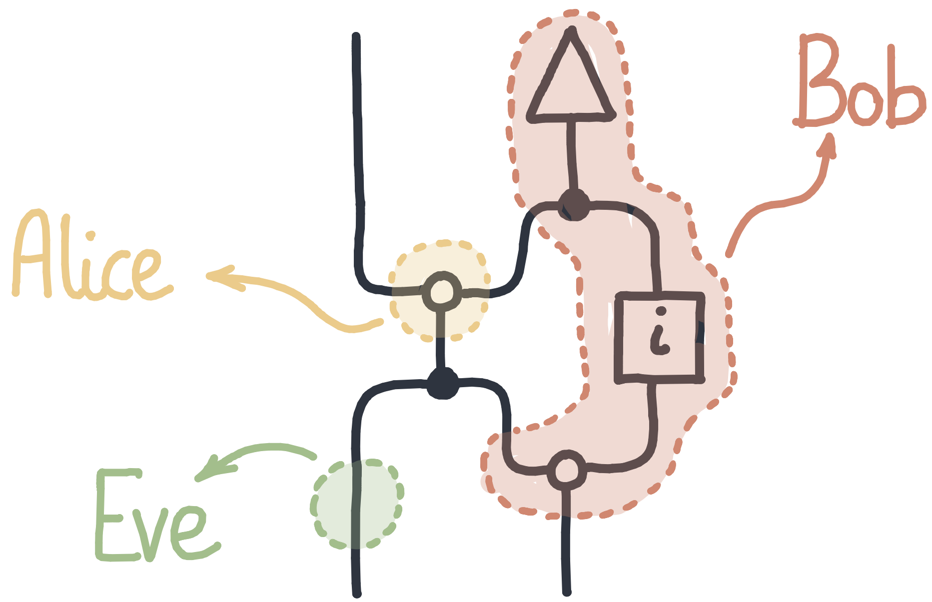
Optics are Universal



Towards Message Theories

Optics are the free normalization of the cofree produoidal over a monoidal category.

- This explains optics as supermaps or data-accessors.
- But optics also decompose communication protocols.



This last part brings us to study message passing.

Part 5: Message Theories

Message Theories

Set of types, representing resource types: $X, Y, Z, \ldots$

Two actions for each resource: send and receive,

$X^\bullet$ means "send X".

X° means "receive X".

Lists of actions represent sequencing of the actions.

$\Gamma = X^\circ, Y^\bullet, Z^\bullet, W^\circ$ means "ask for X; send Y and then Z; finally, receive W".

Message Theories

$$\frac{}{\varepsilon}\,\text{NOP} \qquad \frac{}{X^\circ, X^\bullet}\,\text{SPW} \qquad \frac{\Gamma, X^\bullet, X^\circ, \Delta}{\Gamma, \Delta}\,\text{LNK} \qquad \frac{\Gamma \quad \Delta}{[\Gamma, \Delta]_\sigma}\,\text{SHF}_\sigma$$

- Doing nothing is a session.
- We can create a receive-send "echo" session.
- We can receive what we just sent.
- Events can be interleaved in any order.
- This presents a monoidal multicategory.

MESSAGE THEORIES

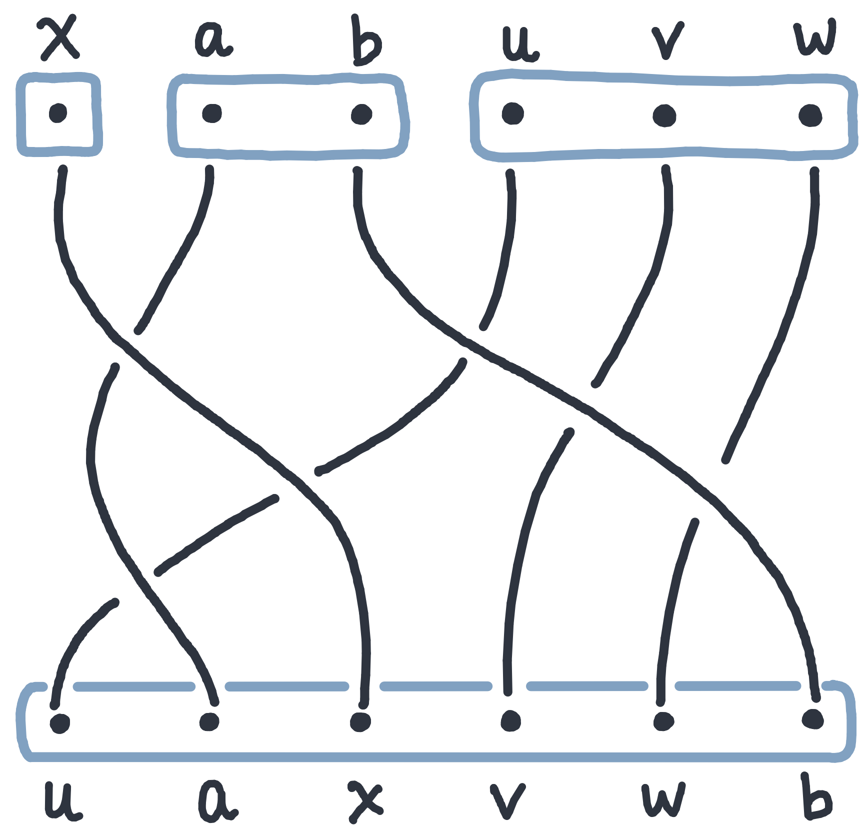
Dualities

Shuffling

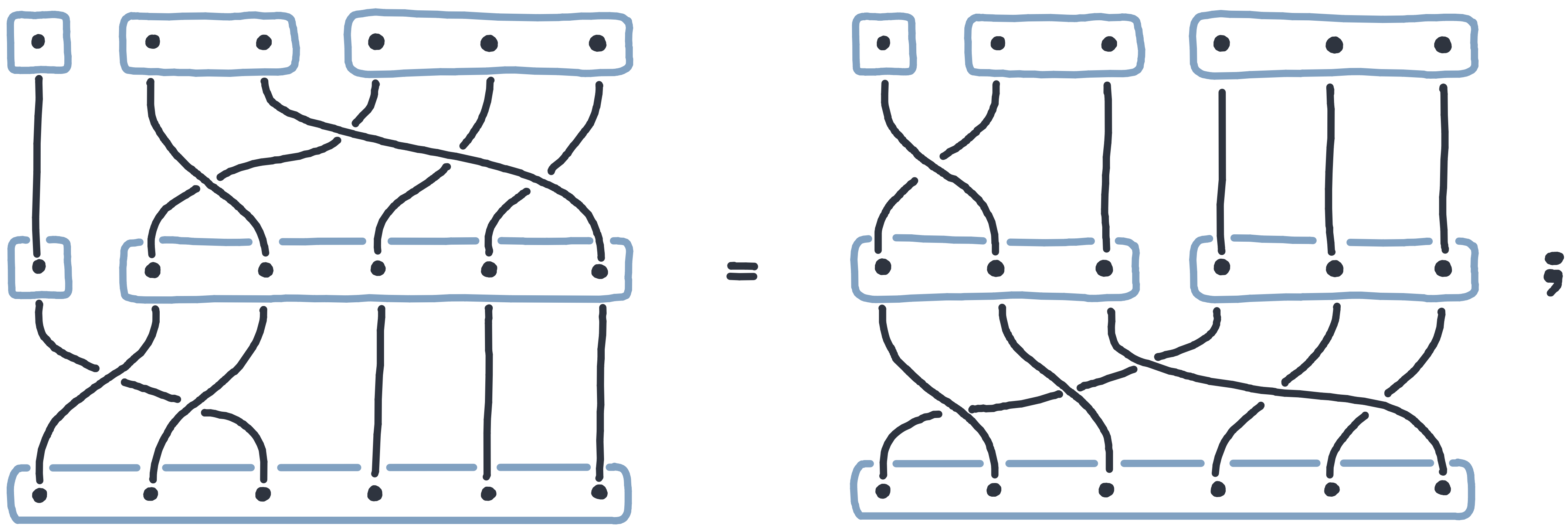
$$\frac{}{\varepsilon}\,\mathrm{NOP} \qquad \frac{\Gamma, \Delta}{\Gamma, X^{\circ}, X^{\bullet}, \Delta}\,\mathrm{SPW} \qquad \frac{\Gamma, X^{\bullet}, X^{\circ}, \Delta}{\Gamma, \Delta}\,\mathrm{LNK} \qquad \frac{\Gamma \qquad \Delta}{[\Gamma, \Delta]_{\sigma}}\,\mathrm{SHF}_{\sigma}$$

- Doing nothing is a session.
- We can create a receive-send "echo" session.
- We can receive what we just sent.
- Events can be interleaved in any order.
- This presents a monoidal multicategory.

Shuffles



Shuffles



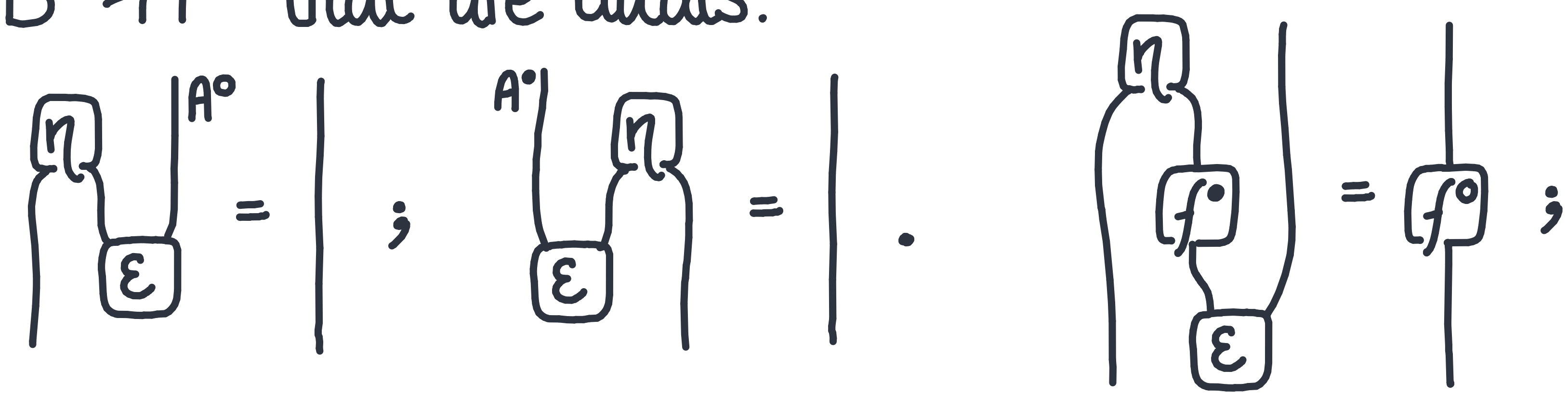
THEOREM. Shuffles form the free physical monoidal multicategory.
PROOF. Consequence of a characterization of the free normal duoidal due to Grabowski and then to Shapiro and Spivak.

POLARIZATION

"Polarization is
left adjoint to taking
left adjoints."

$$\begin{tikzcd} \mathrm{MonCat} \arrow[r, bend left, "\mathrm{LeftAdj}"] \arrow[r, phantom, "\top"] & \mathrm{MonCat} \arrow[l, bend left, "\mathrm{Polar}"] \end{tikzcd}$$

The free polarized monoidal category over a monoidal has a duality $(A^\circ \dashv A^\bullet, \varepsilon_A, \eta_A)$ for each object A, and pair of functions $f^\bullet : A^\bullet \to B^\bullet$ and $f^\circ : B^\circ \to A^\circ$ that are duals.



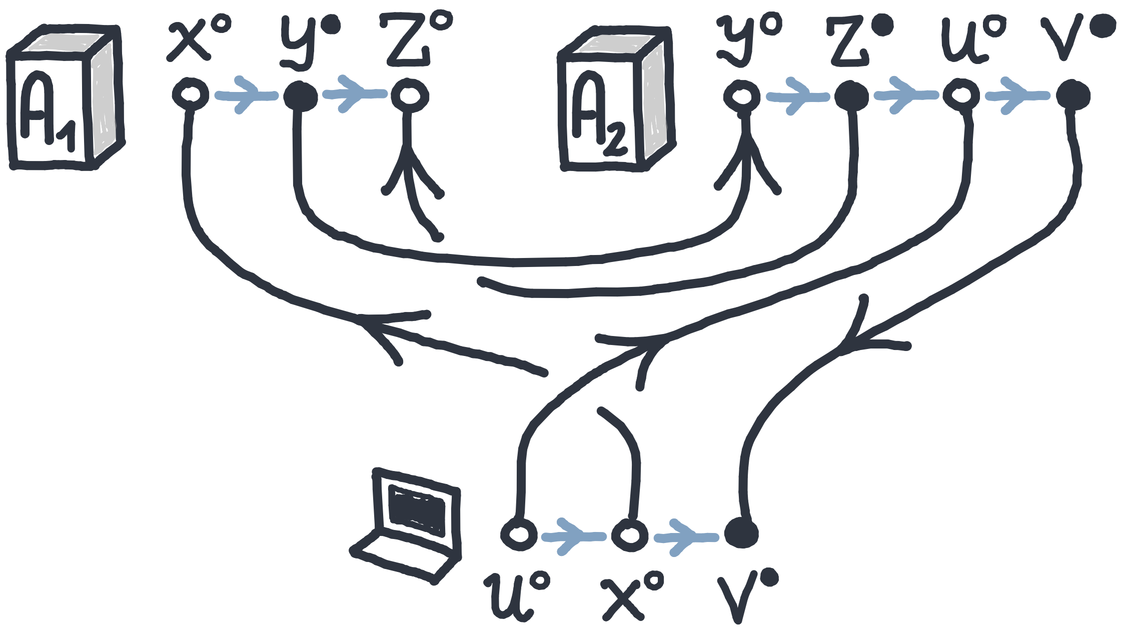
Message Theories

$$\frac{}{\varepsilon}\,\textsf{NOP} \qquad \frac{\Gamma, \Delta}{\Gamma, X^\circ, X^\bullet, \Delta}\,\textsf{SPW} \qquad \frac{\Gamma, X^\bullet, X^\circ, \Delta}{\Gamma, \Delta}\,\textsf{LNK} \qquad \frac{\Gamma \quad \Delta}{[\Gamma, \Delta]_\sigma}\,\textsf{SHF}_\sigma$$

Derivations of the logic of message theories are multimorphisms of the free polarized physical monoidal multicategory over a set of types.

- Can we characterize these?

Polar Shuffles



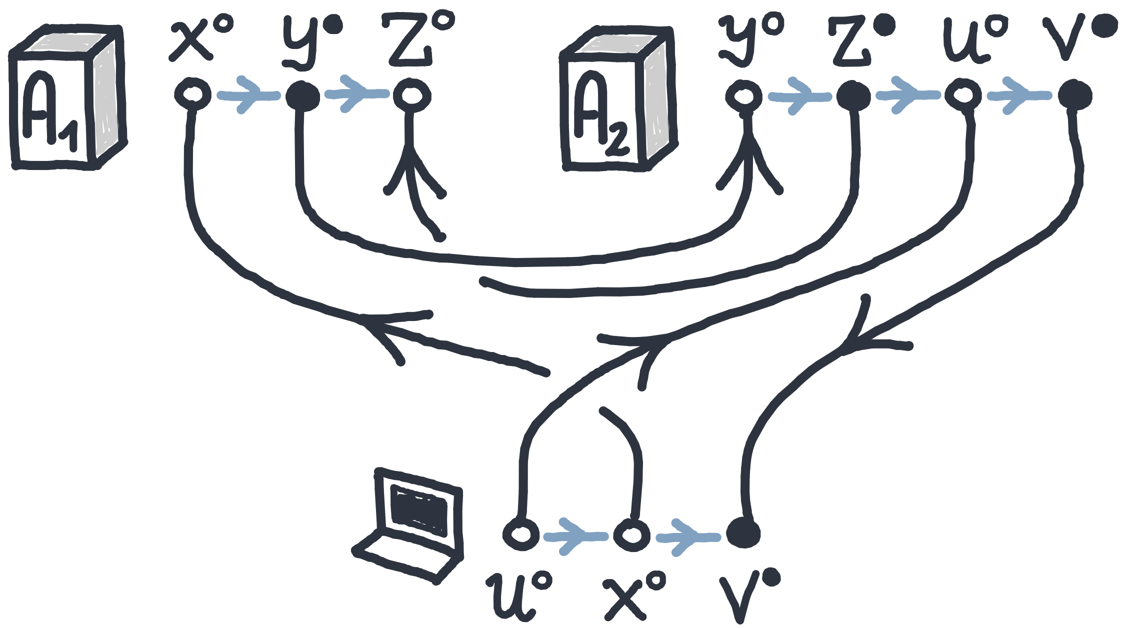
Polar shuffles are bijections

$$A_1^\bullet + \cdots + A_n^\bullet + X^\circ \longrightarrow A_0^\circ + \cdots + A_n^\circ + X^\bullet$$

inducing an acyclic graph.

Polar shuffles coincide with derivations of a message theory.
Polar shuffles form the free polarized physical monoidal multicategory

Polar Shuffles

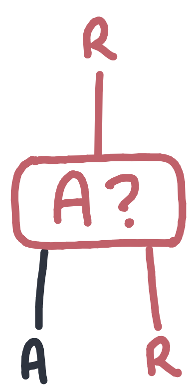



```
Client (u°, x°, v•) {
  A1(x°, y•, z°),
  A2(y°, z•, u°, v•)
}
```

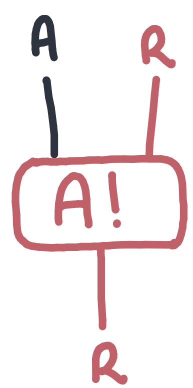
Polar shuffles coincide with derivations of a message theory.
Polar shuffles form the free polarized physical monoidal multicategory

Message Theories vs Process Theories

Sessions of the free message theory over a process theory are string diagrams extended with "send" and "receive" effects.



;



;

for each object $A \in \mathcal{M}_{obj}$

The cofree process theory on a message theory has as morphisms the "receive-then-send" sessions:

$$X^\circ, Y^\bullet.$$

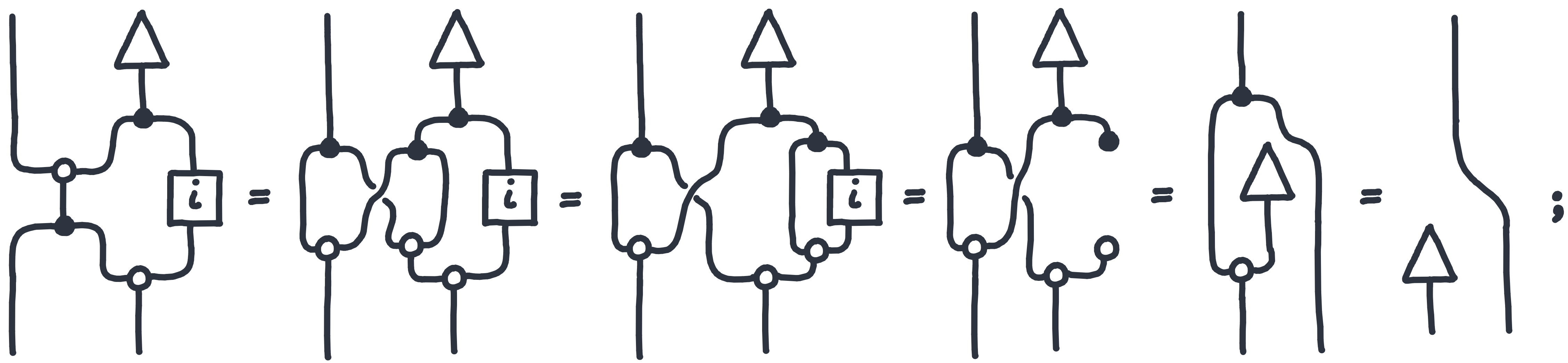
THEOREM.

$$\begin{tikzcd} \mathbf{SymMon} \arrow[r, bend left, "\text{Sessions}"] \arrow[r, phantom, "\perp"] & \mathbf{Msg} \arrow[l, bend left, "\text{Send-Receive}"] \end{tikzcd}$$

PART 6: Examples

One-Time Pad

Broadbent & Karvonen propose a formalization of the one-time pad in a monoidal category with a Hopf algebra with an integral.

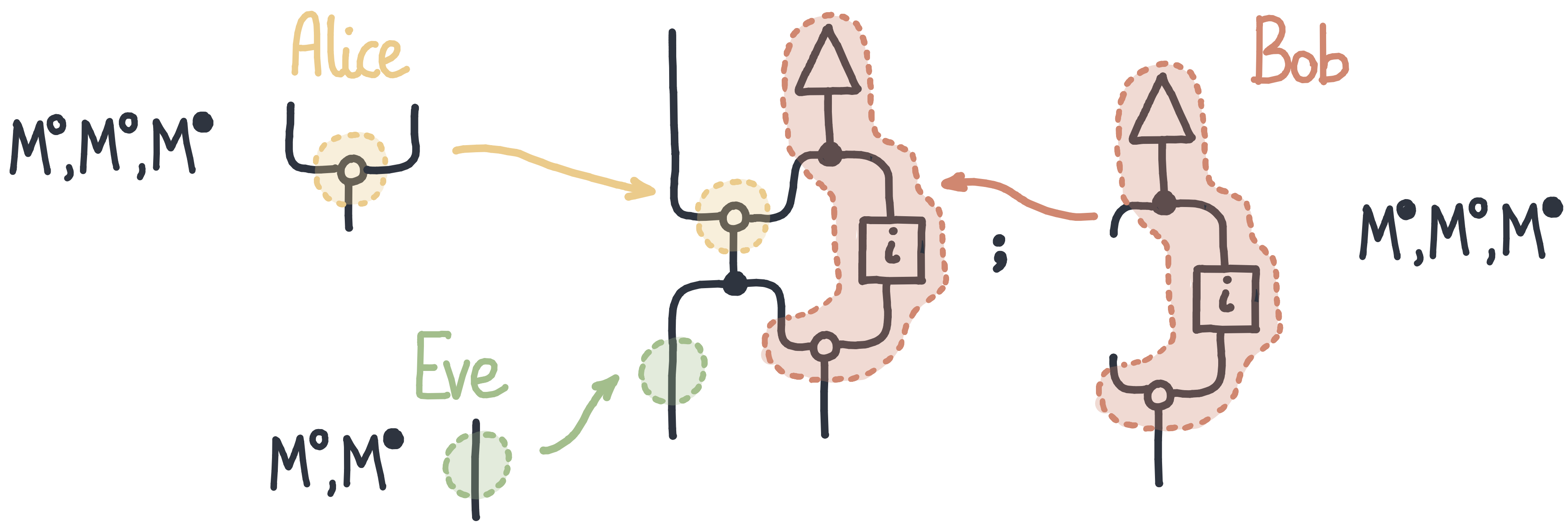


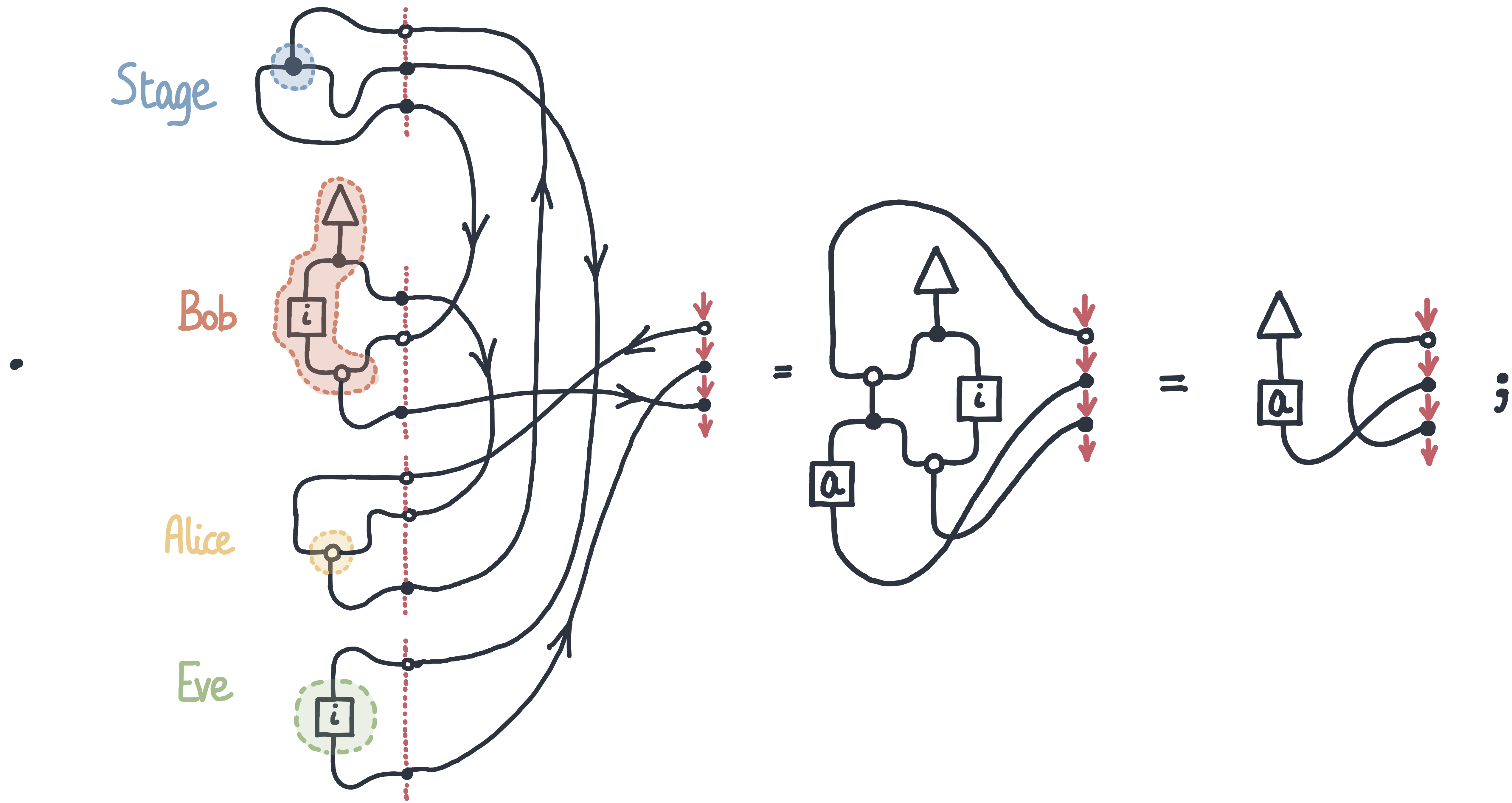
We can reason about security using string diagrams.

Broadbent & Karvonen. Categorical Composable Cryptography.

Multi-Party Processes

We want to split the morphism into different agents: Alice does not control the broadcast; Eve can only attack at the end; Bob keeps a bit in memory.




Stage
Bob
Alice
Eve
i
i
a
i
a

Multi-Party Processes

```
oneTimePad(alice,bob,eve,msg) = do
  key <- bob₀()
  crypt <- alice(msg, key)
  () <- eve(crypt)
  msg <- bob₁(crypt)
  return msg
```

```
alice(msg, key) = do
  crypt <- xor(msg,key)
  return crypt
```

```
eve(crypt) = do
  return crypt
```

```
bob() = do
  key <- randomBit
  !key
  ?crypt
  msg <- xor(crypt,key)
  return msg
```

These allow for code modularity. Send and receive types arise naturally.

github.com/mroman42/produoidal-algebra-code

Next Steps

- **CONJECTURE.** Runtime wire diagrams for Premonoidal & Freyd bicategories work 'as expected'.
  - Paquet, Saville. Strong Pseudomonads and Premonoidal Bicategories.
  - Bartlett. Quasistrict Symmetric Monoidal 2-Categories via Wire Diagrams.
- **CONJECTURE.** Type theory for message-passing over a monoidal mimicks send/receive types.
  We can extend to branching ⊕/& and iteration, in multi-party session using linear actions and feedback.
  - Honda, Yoshida, Carbone. Multiparty Asynchronous Session Types.
  - Cockett, Pastro. The Logic of Message-Passing.
  - Earnshaw, Hefford, Román. Contouring Prostar-Autonomous Categories. UNPUBLISHED.

  The "Markov with conditionals" case is of particular interest: good characterization
- **GAME SEMANTICS.** Vast literature to compare to, outside the original scope.

END

Proof: Polar shuffles are Message derivations

1. Shuffles form the free physical monoidal multicategory. Prop 4.2.9
2. Message theories are shuffles with duals, by definition.
3. Message theories are the free polarized physical monoidal multicategory.
4. Message theories are coherent, by finding their normal form. Thm 4.1.8
5. Polar shuffles are coherent, by definition. Prop 4.4.4
6. A polar shuffle between some types exists iif a message theory derivation exists.
7. Polar shuffles are message theory derivations. Prop 4.4.9
8. Polar shuffles form the free polarized physical mon. multicategory. Thm 4.4.11

PROOF: Sessions and Processes form an adjunction

1. We construct a effectful category of sessions over a strict sym. mon. cat. Def 4.5.3
2. Sessions form a message theory, by topology. Prop 4.5.6
3. Sessions are combs. Prop 4.5.5
4. There exists a unit inProc: $\mathbb{C} \to$ PROC(SESSION($\mathbb{C}$)). Lem. 4.5.8
5. Sessions and Processes form an adjunction. Thm 4.5.9

Proof: String diagrams for effectful categories

1. Braiding runtime forms cliques.
2. Braid cliques on the runtime monoidal form a effectful, $\mathsf{Eff}(\mathbb{V},\mathbb{G})$. Lem 1.7.5
3. There exists an id. on objs. $\mathsf{Mon}(\mathbb{V}) \to \mathsf{Eff}(\mathbb{V},\mathbb{G})$ preserving mon. structure. Lem 1.7.6
4. There exists a unique effectful functor out of $\mathsf{Mon}(\mathbb{V}) \to \mathsf{Eff}(\mathbb{V},\mathbb{G})$. Lem 1.7.7
5. The free effectful has morphisms $A \to B$ the $R \otimes A \to R \otimes B$ of the runtime monoidal. Thm 1.7.8
6. String diagrams with runtime are a language for effectful categories. Cor 1.7.9

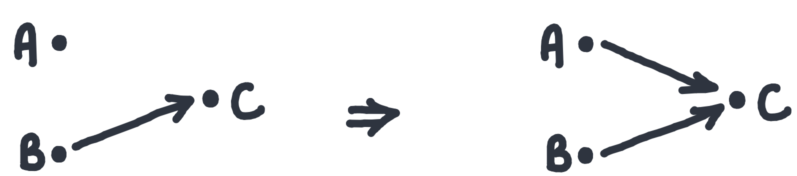
Duoidal Categories

Duoidal categories are not coherent, there are two formal maps

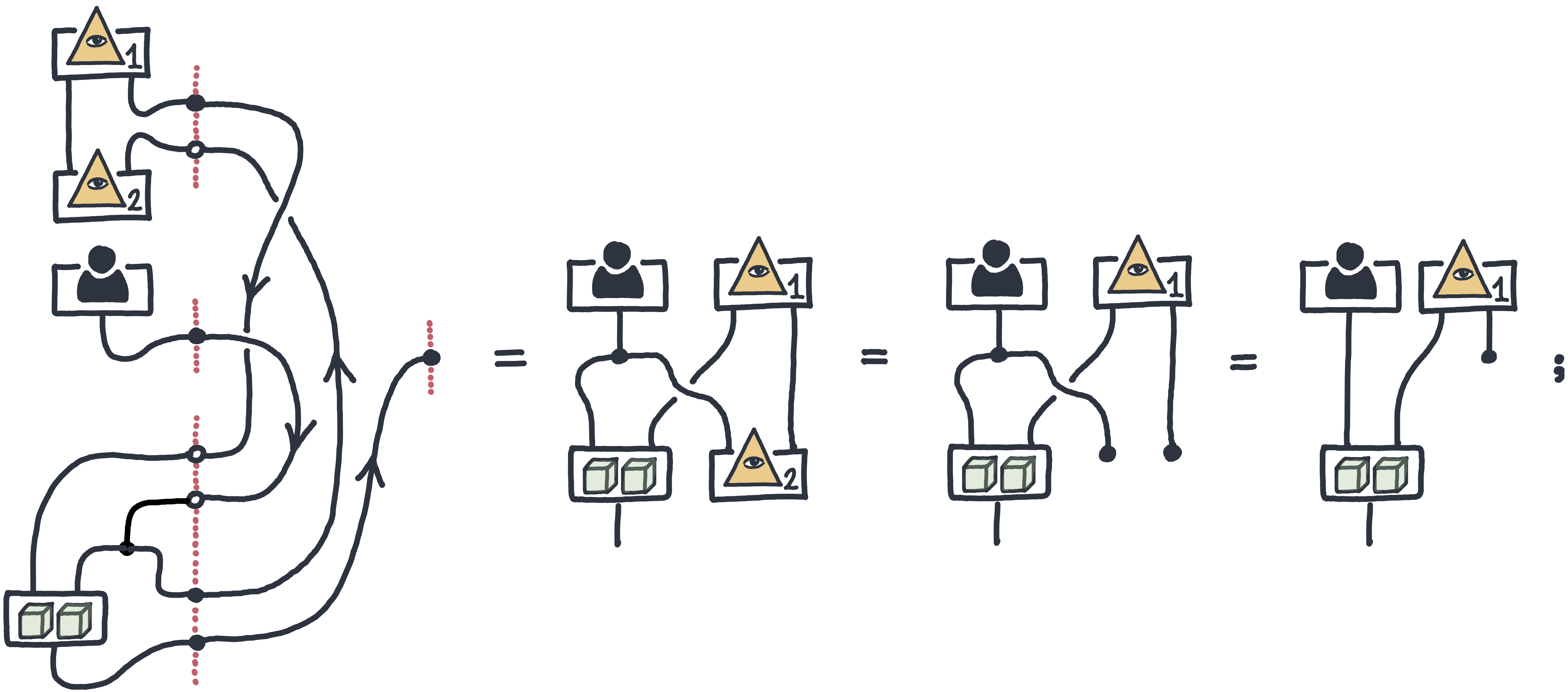
$$I \triangleleft I \to I.$$

However, physical duoidal categories are coherent: the free physical duoidal over a set of objects is a full subcategory of poset inclusions. There is at most a single morphism between any two objects where every type appears exactly once with each variance.

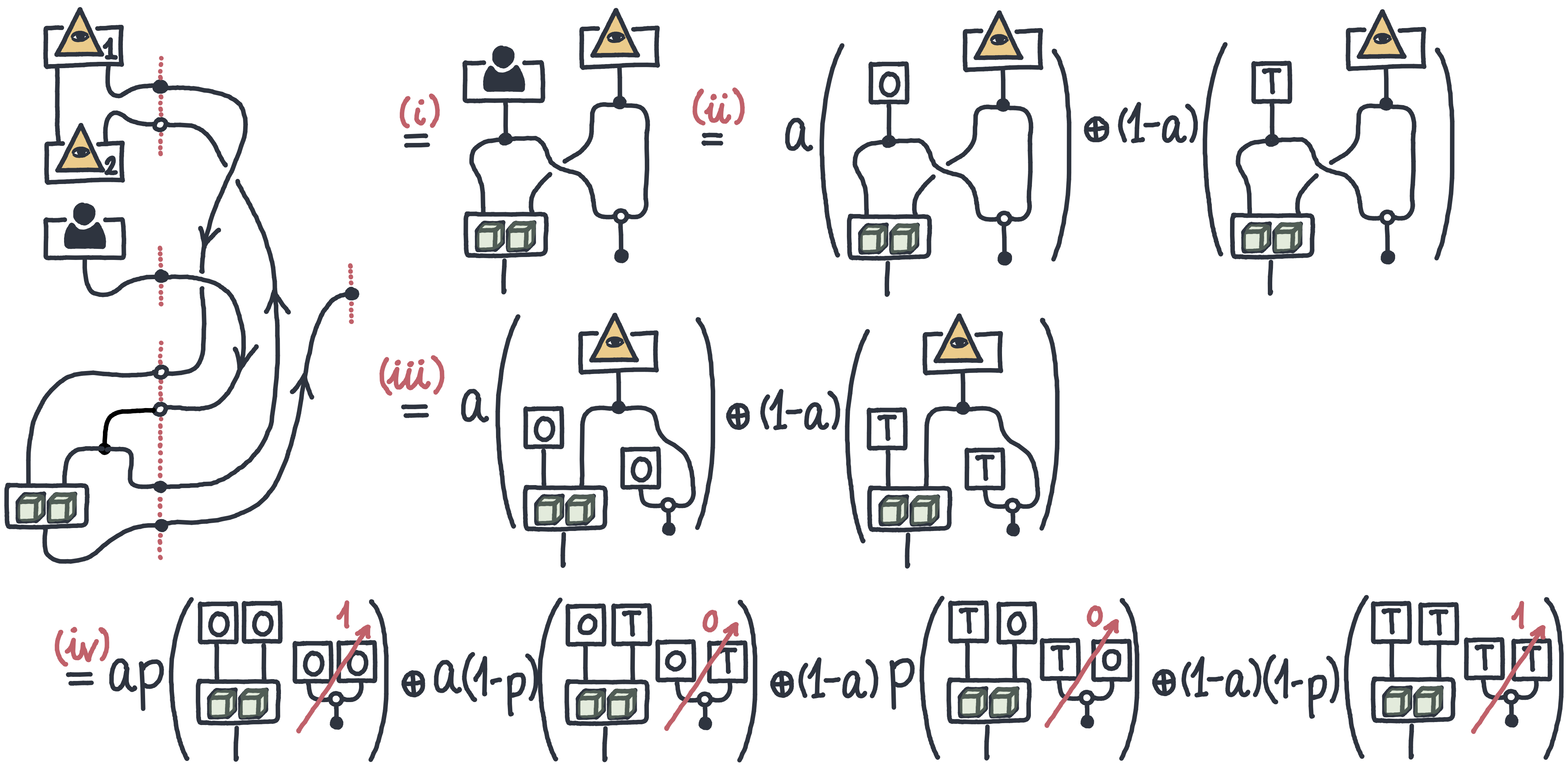
$$\begin{tikzcd} A \otimes (B \triangleleft C) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\shortparallel"] & (A \otimes B) \triangleleft C. \end{tikzcd}$$



Grabowski '81
Gischer '88



1
2
1
2
1
1



$$\overset{(v)}{=}\ ap(1000\text{€}) \oplus (1-a)\cancelto{\varepsilon}{(1-p)}(1\text{€}) \ \overset{(vi)}{\leq}\ p(1000\text{€}).$$

Message Theories

$$\frac{}{\varepsilon}\,\text{NOP} \qquad \frac{\Gamma, \Delta}{\Gamma, X^{\circ}, X^{\bullet}, \Delta}\,\text{SPW} \qquad \frac{\Gamma, X^{\bullet}, X^{\circ}, \Delta}{\Gamma, \Delta}\,\text{LNK} \qquad \frac{\Gamma \qquad \Delta}{[\Gamma, \Delta]_{\sigma}}\,\text{SHF}_{\sigma}$$

- Doing nothing is a session.
- We can create a receive-send "echo" session.
- We can receive what we just sent.
- Events can be interleaved in any order.
- This presents a **monoidal multicategory**.

SHUFFLES

The physical monoidal multicategory of shuffles over an alphabet Σ has

- objects the words of Σ^*;
- mutimorphisms $\mathsf{Shuf}(w_1, \ldots, w_n ; w)$ are shufflings into w.

This is not posetal, in fact,

$$\#\mathsf{Shuf}(a^{n_1}, \ldots, a^{n_k} ; a^{n_1+\cdots+n_k}) = \frac{(n_1+\cdots+n_k)!}{n_1! \cdots n_k!} .$$

However, if each letter appears exactly once, then the suffle, if it exists, it is unique.

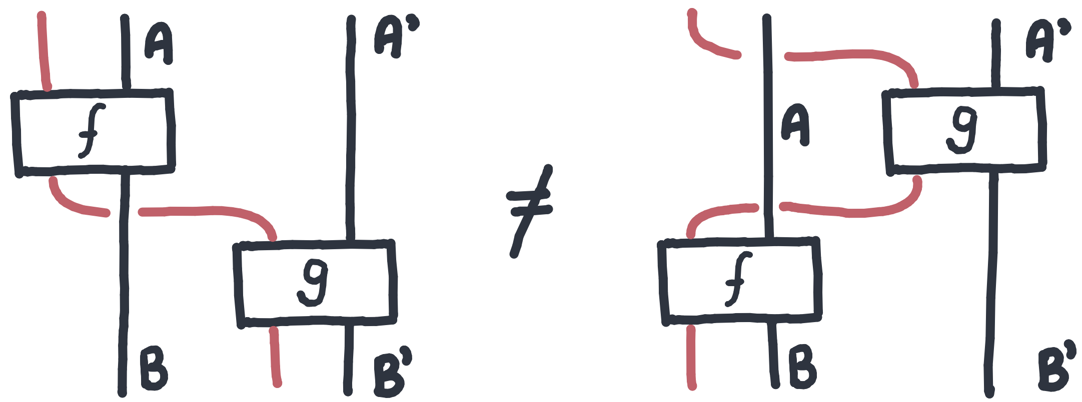
EXAMPLE. Unique possible shuffle $xy, wz \to xwyz$.

Next

$$
\begin{tikzcd}[column sep=huge]
\text{Monoidal Category} \arrow[r, bend left, "\text{Mon. Context}"] \arrow[r, phantom, "\top"] \arrow[d] & \text{Produoidal Category} \arrow[l, bend left, "\text{Mon. Contour}"] \arrow[d] & \\
\text{Category} \arrow[r, bend left, "\text{Context}"] \arrow[r, phantom, "\top"] & \text{Promonoidal Category} \arrow[l, bend left, "\text{Contour}"] & \text{Duoidal Category} \arrow[ul, dashed]
\end{tikzcd}
$$

PreMonoidal Categories

Premonoidal categories extend monoidal categories with effects.

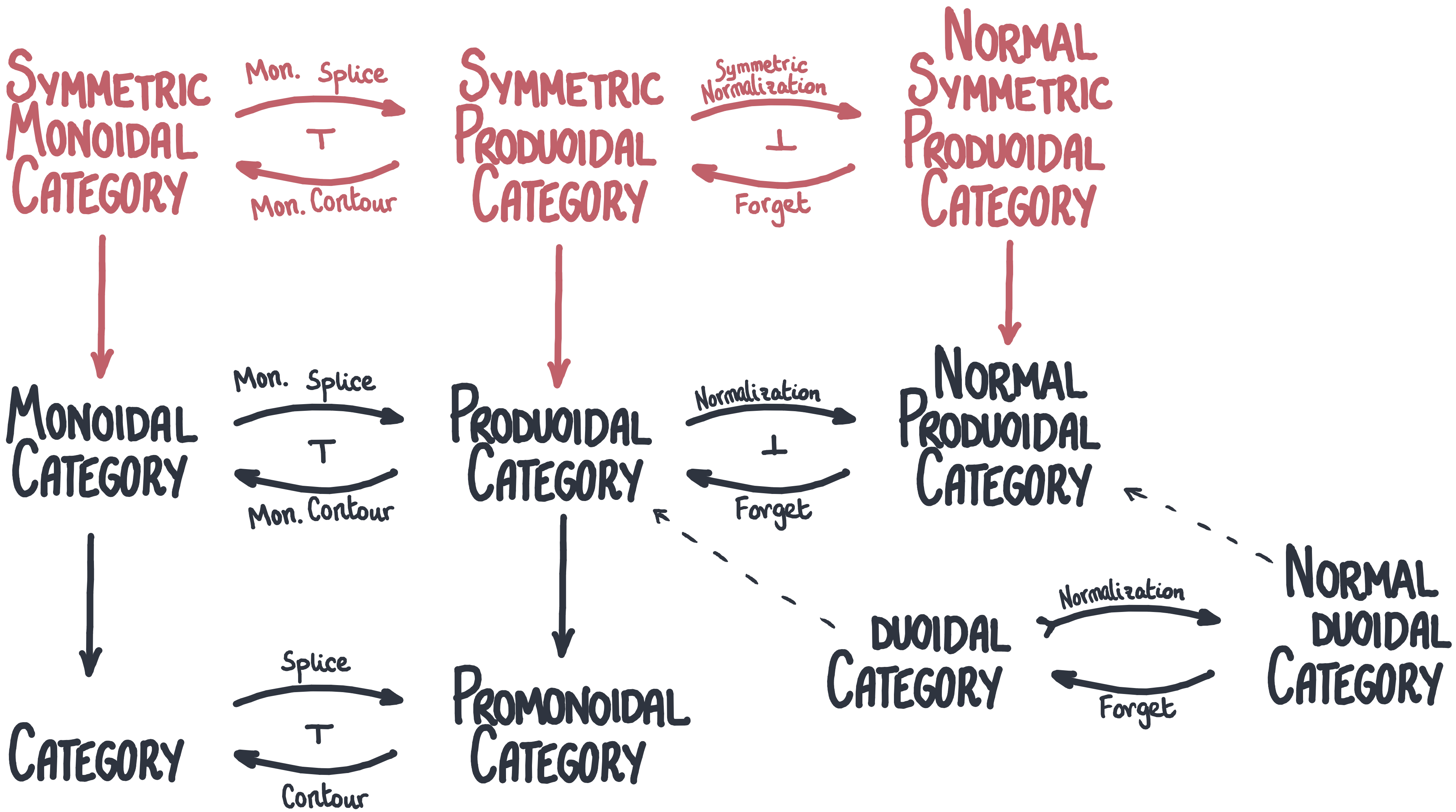


Failure of Interchange

THEOREM. String diagrams with runtime are the internal language of premonoidal categories.

Jeffrey Román

NEXT



Symmetric Monoidal Category
Mon. Splice
⊤
Mon. Contour
Symmetric Produoidal Category
Symmetric Normalization
⊥
Forget
Normal Symmetric Produoidal Category
Monoidal Category
Mon. Splice
⊤
Mon. Contour
Produoidal Category
Normalization
⊥
Forget
Normal Produoidal Category
Category
Splice
⊤
Contour
Promonoidal Category
Duoidal Category
Normalization
Forget
Normal Duoidal Category

MONOIDAL CATEGORIES: PROCESS THEORIES

Monoidal categories are coherent: there is at most a single morphism between any two objects in the *free* monoidal category over some objects.

$$\begin{tikzcd} A \otimes (B \otimes C) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "="] & (A \otimes B) \otimes C \end{tikzcd}$$

Symmetric monoidal categories do not satisfy the same property.

$$\begin{tikzcd} A \otimes A \arrow[r, bend left, "id"] \arrow[r, bend right, "swap"'] \arrow[r, phantom, "\neq"] & A \otimes A \end{tikzcd}$$

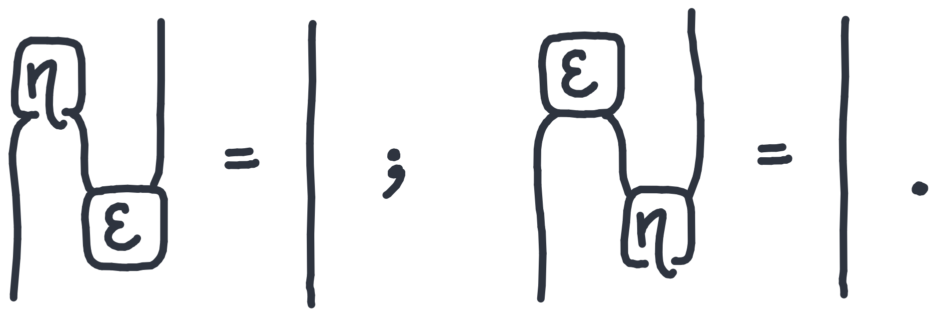
Still, they satisfy another coherence property: there is at most a single morphism between any two objects where every type appears exactly once with each variance.

$$\begin{tikzcd} A \otimes (B \otimes C) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "="] & B \otimes (C \otimes A) \end{tikzcd}$$

Polarization

The second ingredient for message passing is the duality SEND/RECEIVE.

A duality is a pair of objects with two morphisms $(L \dashv R, \varepsilon, \eta)$ such that



"Polarization is left adjoint to taking left adjoints."

The free polarized monoidal category over a monoidal h

$$\begin{tikzcd} \mathrm{MonCat} \arrow[r, bend left, "\mathrm{LeftAdj}"] \arrow[r, phantom, "\perp"] & \mathrm{MonCat} \arrow[l, bend left, "\mathrm{Polar}"] \end{tikzcd}$$

Message Theories: Axioms

Linking is natural with respect to shufflings.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, m_1\\ \Gamma_1, X^\bullet, X^\circ, \Gamma_2\end{array}}{\Gamma_1, \Gamma_2} \qquad \begin{array}{c}\vdots\, m_2\\ \Delta_1, \Delta_2\end{array}}{[\Gamma_1;\Delta_1]_\sigma, [\Gamma_2;\Delta_2]_\tau} \;=\; \dfrac{\dfrac{\begin{array}{c}\vdots\, m_1\\ \Gamma_1, X^\bullet, X^\circ, \Gamma_2\end{array} \qquad \begin{array}{c}\vdots\, m_2\\ \Delta_1, \Delta_2\end{array}}{[\Gamma_1;\Delta_1]_\sigma, X^\bullet, X^\circ, [\Gamma_2;\Delta_2]_\tau}}{[\Gamma_1;\Delta_1]_\sigma, [\Gamma_2;\Delta_2]_\tau}\;;$$

Spawning is natural with respect to shufflings.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, m_1\\ \Gamma_1, \Gamma_2\end{array}}{\Gamma_1, X^\circ, X^\bullet, \Gamma_2} \qquad \begin{array}{c}\vdots\, m_2\\ \Delta_1, \Delta_2\end{array}}{[\Gamma_1;\Delta_1]_\sigma, X^\circ, X^\bullet, [\Gamma_2;\Delta_2]_\tau} \;=\; \dfrac{\dfrac{\begin{array}{c}\vdots\, m_1\\ \Gamma_1, \Gamma_2\end{array} \qquad \begin{array}{c}\vdots\, m_2\\ \Delta_1, \Delta_2\end{array}}{[\Gamma_1;\Delta_1]_\sigma, [\Gamma_2;\Delta_2]_\tau}}{[\Gamma_1;\Delta_1]_\sigma, X^\circ, X^\bullet, [\Gamma_2;\Delta_2]_\tau}\;;$$

MESSAGE THEORIES: AXIOMS

Shuffles compose as in the multicategory of shuffles.

$$\dfrac{\dfrac{\overset{\vdots\, m_1}{\Gamma_1} \qquad \overset{\vdots\, m_2}{\Gamma_2}}{[\Gamma_1;\Gamma_2]_\sigma} \qquad \overset{\vdots\, m_3}{\Gamma_3}}{[[\Gamma_1;\Gamma_2]_\sigma;\Gamma_3]_\tau} \overset{(1A)}{=} \dfrac{\overset{\vdots\, m_1}{\Gamma_1} \qquad \dfrac{\overset{\vdots\, m_2}{\Gamma_2} \qquad \overset{\vdots\, m_3}{\Gamma_3}}{[\Gamma_2;\Gamma_3]_{\tau'}}}{[\Gamma_1;[\Gamma_2;\Gamma_3]_{\tau'}]_{\sigma'}} \;;$$

$$\dfrac{\overset{\vdots\, m}{\Gamma_1} \qquad \overline{}}{[\Gamma_1;]_*} \overset{(1B)}{=} \vdots\, m \;;$$

$$\dfrac{\overset{\vdots\, m_1}{\Gamma} \qquad \overset{\vdots\, m_2}{\Delta}}{[\Gamma;\Delta]_\sigma} \overset{(1C)}{=} \dfrac{\overset{\vdots\, m_2}{\Delta} \qquad \overset{\vdots\, m_1}{\Gamma}}{[\Gamma;\Delta]_{\tilde{\sigma}}} \;;$$

This is the free normal monoidal multicategory on a set of types.

Message Theories: Axioms

Spawning and linking are duals.

$$\dfrac{\dfrac{\dfrac{\vdots\, m}{\Gamma, X^\circ, \Delta}}{\Gamma, X^\circ, X^\bullet, X^\circ, \Delta}}{\Gamma, X^\circ, \Delta} \overset{(4_L)}{=} \vdots\, m \quad ; \qquad \dfrac{\dfrac{\dfrac{\vdots\, m}{\Gamma, X^\bullet, \Delta}}{\Gamma, X^\bullet, X^\circ, X^\bullet, \Delta}}{\Gamma, X^\bullet, \Delta} \overset{(4_R)}{=} \vdots\, m \quad ;$$

Spawning and linking interchange.

$$\dfrac{\dfrac{\dfrac{\vdots\, m}{\Gamma_1, X^\bullet, X^\circ, \Gamma_2, \Gamma_3}}{\Gamma_1, X^\bullet, X^\circ, \Gamma_2, Y^\circ, Y^\bullet, \Gamma_3}}{\Gamma_1, \Gamma_2, Y^\circ, Y^\bullet, \Gamma_3} \overset{(5_A)}{=} \dfrac{\dfrac{\dfrac{\vdots\, m}{\Gamma_1, X^\bullet, X^\circ, \Gamma_2, \Gamma_3}}{\Gamma_1, \Gamma_2, \Gamma_3}}{\Gamma_1, \Gamma_2, Y^\circ, Y^\bullet, \Gamma_3} \quad ;$$

This is the free polarized normal monoidal multicategory on a set of types.