Monoidal Profunctors

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MOTIVATION

```
class MONOIDAL P where
  parallel :: P a b → P c d → P (a⊗b) (c⊗d)
  nothing :: P I I
```

Where is this a monoid? Are these the correct notion of profunctor between monoidal categories?

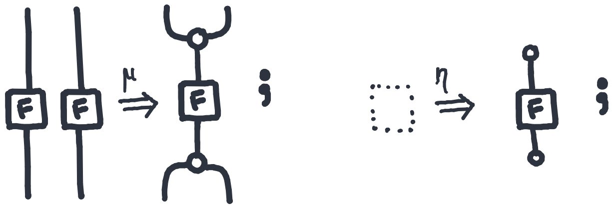


FIG 1. Monoidal profunctor in PROF.

PSEUDOMONOIDS

DEFINITION (Pseudomonoid). Let $\mathbb{B}, \otimes$ be a monoidal bicategory. A **pseudomonoid** is a 0-cell A together with 1-cells $M : A \otimes A \to A$ and $U : I \to A$; and together with 2-cells representing unitors and associators.

All formal equations with α, λ, ρ hold true.

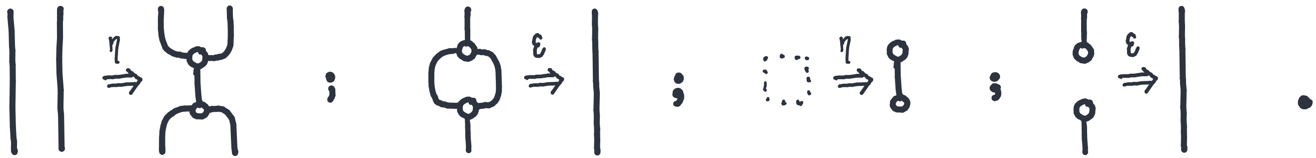
EXAMPLE. Pseudomonoids in $\mathbf{CAT}, \times$ are monoidal categories. Pseudomonoids in $\mathbf{CAT}, \square$ (with the funny tensor product) should be premonoidal categories (has anyone proven this?).

Pseudomonoids in a monoidal category are monoids.

Map Pseudomonoids

MOTIVATION. A monoidal category $\mathbb{C}, \otimes$ gives rise to a pseudomonoid-pseudocomonoid adjoint pair in **PROF**, with the representable and corepresentable profunctors for the tensor and unit.

DEFINITION (Map Pseudomonoid). A map pseudomonoid is an adjoint pair of a pseudomonoid-pseudocomonoid.



All equations between η, ε and the coherence 2-cells hold true.

MONOIDAL TWISTED ARROW

DEFINITION (Twisted arrow mon.cat). Let $\mathbb{C}, \otimes$ be a monoidal category. The twisted arrow category $\mathrm{Tw}\,\mathbb{C}$ is a monoidal category where

- objects are arrows $f : A \to B$ in $\mathbb{C}$;
- morphisms are commuting twisted squares ;
- the unit is $id : I \to I$;
- the tensor is the tensor of arrows.

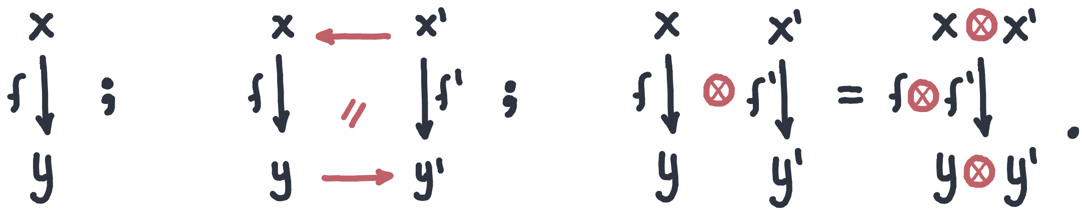
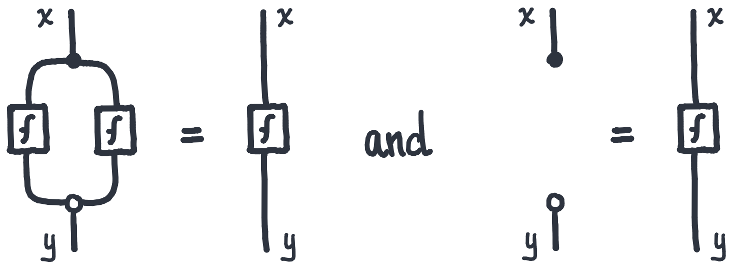


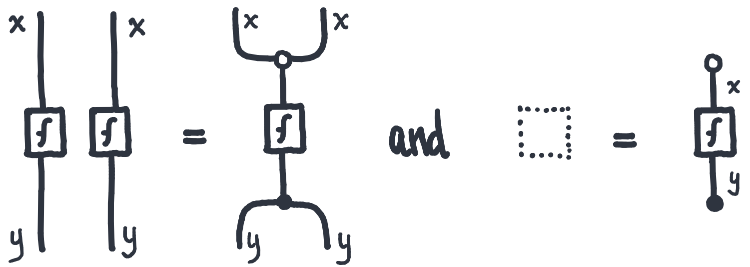
FIG 1. Objects, morphisms and tensor in $\mathrm{Tw}\,\mathbb{C}$.

MONOIDAL TWISTED ARROW

DEFINITION (Twisted monoid). A twisted monoid is a monoid in $\mathrm{Tw}\,\mathbb{C}$. Explicitly, it is an arrow $f : x \to y$ where x is a comonoid, y is a monoid, and the following equations hold.



These are uninteresting. Just a pair monoid/comonoid. Can we laxate everything?



The "dual" are twisted comonoids. These seem more interesting. Any examples?

Monoidal Twisted Arrow Bicategory

DEFINITION (Twisted arrow mon. bicat.). Let $\mathbb{C}, \otimes$ be a monoidal bicategory. The twisted arrow bicategory $\mathrm{Tw}\,\mathbb{C}$ is a monoidal bicategory where

- 0-cells are arrows $f : A \to B$ in $\mathbb{C}$;
- 1-cells are lax twisted squares;
- the unit is $\mathrm{id} : I \to I$;
- the tensor is the tensor of arrows;
- 2-cells are cylindrical fillings.

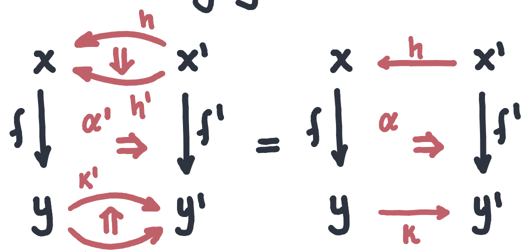


FIG 2. 2-cells in $\mathrm{Tw}\,\mathbb{C}$.

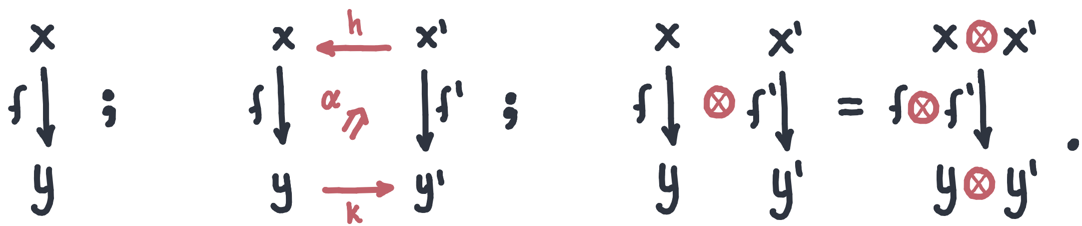
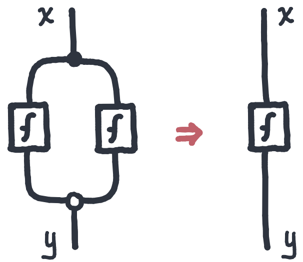


FIG 1. Objects, morphisms and tensor in $\mathrm{Tw}\,\mathbb{C}$.

Monoidal Twisted Arrow Bicategory

DEFINITION (Twisted pseudomonoid). A twisted pseudomonoid is a pseudomonoid in $\mathbf{Tw}\,\mathbb{C}$. Explicitly, it is an arrow $f : x \to y$ where x is a pseudocomonoid, y is a pseudomonoid, and the following equations hold.



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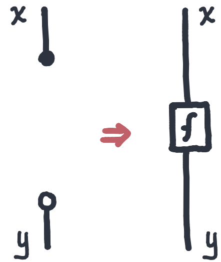
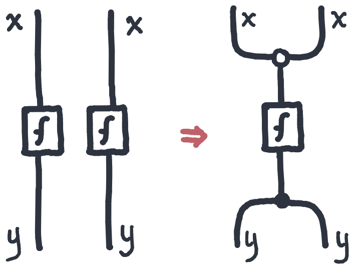


Lax twisted pseudomonoids. These are also called convolution monoids.



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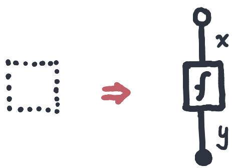
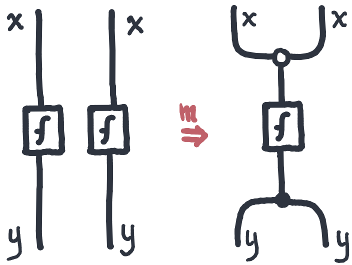


The "dual" are colax twisted comonoids. These are monoidal profunctors. Any examples?

Colax Twisted Comonoids

An arrow and a pair of 2-cells for multiplication and unit.



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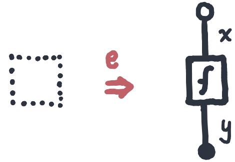
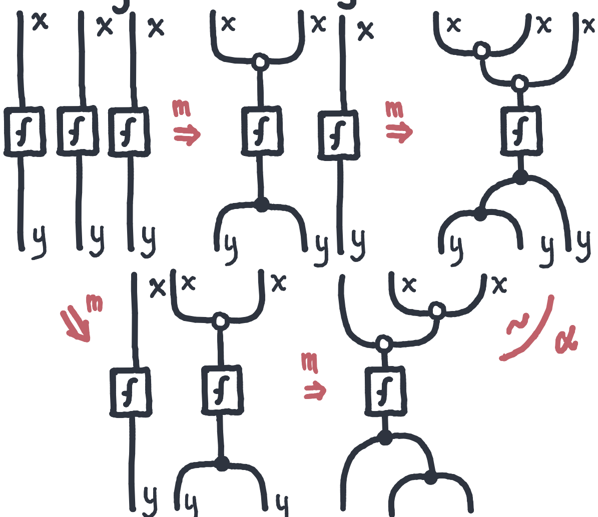
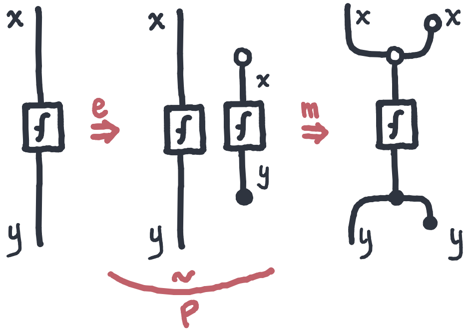


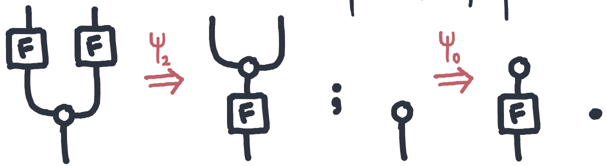
Associativity and unitality become the following invertible 2-cells.





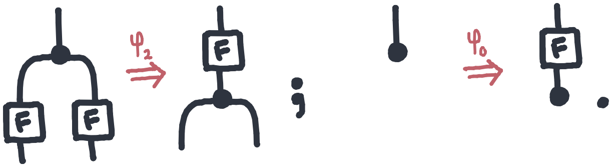
Maps of Map Pseudomonoids

Proposition. Let $(A, \curlyvee, \circ, \curlywedge, \circ)$ and $(B, \curlyvee, \bullet, \curlywedge, \bullet)$ be two map pseudomonoids. Let $F : A \to B$ be a lax homomorphism of pseudomonoids (1).

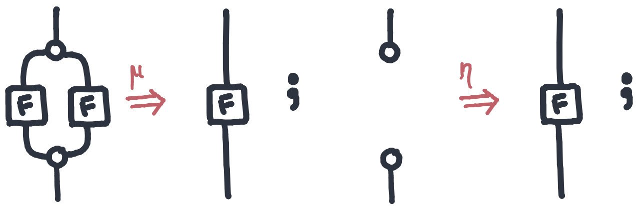


(1)

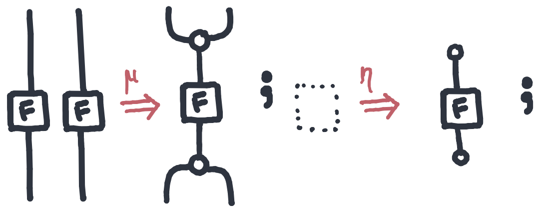
By the mates correspondence, (1) is the same as a colax homomorphism of pseudomonoids (2), a convolution monoid (3) and a colax-twisted comonoid (4).



(2)



(3)



(3)

This is work in progress.

- Any ideas?
- How to make the diagrams formal?