Picturing multivariable adjunctions.

And the 2-Chu construction.

TallCat Chu Construction Seminar

MARIO ROMÁN.

Multivariable Adjunction

An (n,m)-multivariable adjunction $(A_1,\dots,A_n) \to (B_1,\dots,B_m)$ is a profunctor $P : A_1^{op} \times \cdots \times A_n^{op} \times B_1 \times \cdots \times B_m \to \mathrm{Set}$ that is representable in each variable.

$$P(a_1, \dots)$$

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That means that there exist functors $g_1,\dots,g_n$ and $f_1,\dots,f_m$ such that there is the following isomorphism clique.

$$
\begin{aligned}
P(a_1,\dots,a_n,b_1,\dots,b_m) &\cong \hom(a_i, g_i(a_1,\overset{\overset{i}{\downarrow}}{\dots},a_n,b_1,\dots,b_m)) \\
&\cong \hom(f_j(a_1,\dots,a_n,b_1,\overset{\overset{j}{\downarrow}}{\dots},b_m), b_j)
\end{aligned}
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missing a_i!

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\end{aligned}
$$

MULTIVARIABLE ADJUNCTION

- A $(1,1)$-adjunction $A \to B$ is an ordinary adjunction. $\hom(a, g(b)) \cong \hom(f(a), b)$.
- A $(2,1)$-adjunction $A_1, A_2 \to B$ is a triple of functors
$$\begin{aligned} &\hom(f(a_1,a_2), b) \cong \\ &\hom(a_1, g_1(a_2, b_2)) \cong \\ &\hom(a_2, g_2(a_1, b_2)). \end{aligned}$$
- A $(0,1)$-adjunction is an object.

Polycategory of multivariable adjunctions.

Let $P : \Gamma \nrightarrow A, \Delta$ and $Q : A, \Gamma' \nrightarrow \Delta'$ be multivariable adjunctions. We claim that

$$(Q \circ_A P)(\Gamma, \Gamma'; \Delta, \Delta') := \int^{a \in A} P(\Gamma; \Delta, a) \times Q(a, \Gamma'; \Delta')$$

is a multivariable adjunction.

Polycategory of multivariable adjunctions.

Let $P : \Gamma, A \to \Delta$ and $Q : A, \Gamma' \to \Delta'$ be multivariable adjunctions. We claim that

$$(Q \circ_A P)(\Gamma, \Gamma' ; \Delta, \Delta') := \int^{a \in A} P(\Gamma ; \Delta, a) \times Q(a, \Gamma' ; \Delta')$$

is a multivariable adjunction.

What does it mean for P and Q to be representable?

$$\begin{aligned} P(\Gamma ; a, \Delta) &\cong \hom(\Gamma_i, g_i(\Gamma_{\neq i}, a, \Delta)) \\ &\cong \hom(f_j(\Gamma, a, \Delta_{\neq j}), \Delta_j) \\ &\cong \hom(h(\Gamma, \Delta), a) \end{aligned}$$

$$\begin{aligned} Q(\Gamma', a ; \Delta') &\cong \hom(\Gamma'_i, g'_i(\Gamma'_{\neq i}, a, \Delta')) \\ &\cong \hom(f'_j(\Gamma', a, \Delta'_{\neq j}), \Delta'_j) \\ &\cong \hom(a, h'(\Gamma, \Delta)). \end{aligned}$$

Polycategory of multivariable adjunctions.

Let $P : \Gamma, A \to \Delta$ and $Q : A, \Gamma' \to \Delta'$ be multivariable adjunctions. We claim that

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is a multivariable adjunction.

Proof. Let us show it is representable in Γ_i, the rest is analogous.

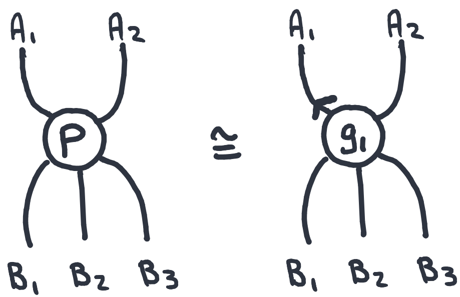
$$\int^{a \in A} P(\Gamma_i, \Gamma_{\neq i} ; a, \Delta) \times Q(a, \Gamma' : \Delta') \cong$$

$$\int^{a \in A} \hom(\Gamma_i, g_i(\Gamma_{\neq i}, a, \Delta)) \times \hom(a, h'(\Gamma', \Delta')) \cong$$

$$\hom(\Gamma_i, g_i(\Gamma_{\neq i}, h'(\Gamma', \Delta'), \Delta))$$

Graphical Calculus.

A representable profunctor is depicted by its representing functor together with an arrowtip pointing in the direction of the represented variable.



Here $P : A_1^{op} \times A_2^{op} \times B_1 \times B_2 \times B_3 \to SET$ is representable in A_1

$$P(a_1, a_2, b_1, b_2, b_3) \cong$$
$$\hom(a_1, g_1(a_2, b_1, b_2, b_3)).$$

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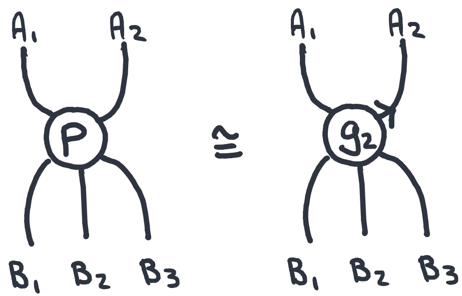
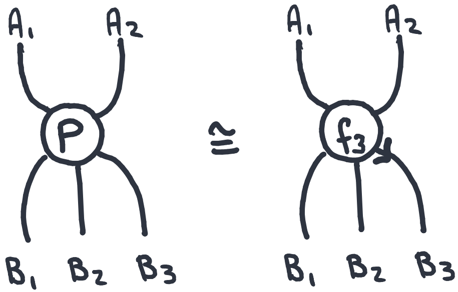


Here $P : A_1^{op} \times A_2^{op} \times B_1 \times B_2 \times B_3 \to SET$ is representable in A_2

$$P(a_1, a_2, b_1, b_2, b_3) \cong$$
$$\hom(a_2, g_2(a_1, b_1, b_2, b_3)).$$

Graphical Calculus.

A representable profunctor is depicted by its representing functor together with an arrowtip pointing in the direction of the represented variable.



Here $P: A_1^{op} \times A_2^{op} \times B_1 \times B_2 \times B_3 \to SET$ is representable in B_3

$$P(a_1, a_2, b_1, b_2, b_3) \cong$$
$$\hom(f_3(a_1, a_2, b_1, b_2), b_3).$$

Graphical Calculus.

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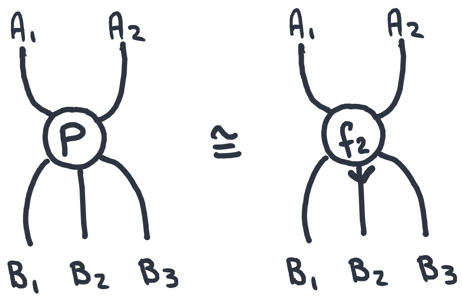
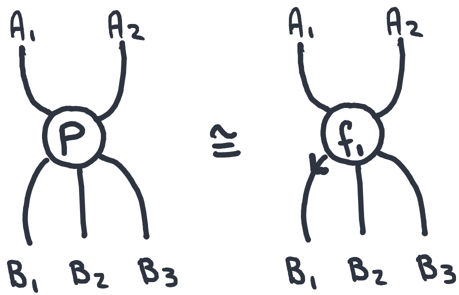


Here $P : A_1^{op} \times A_2^{op} \times B_1 \times B_2 \times B_3 \to SET$ is representable in B_2

$$P(a_1, a_2, b_1, b_2, b_3) \cong \hom(f_2(a_1, a_2, b_1, b_3), b_2).$$

Graphical Calculus.

A representable profunctor is depicted by its representing functor together with an arrowtip pointing in the direction of the represented variable.

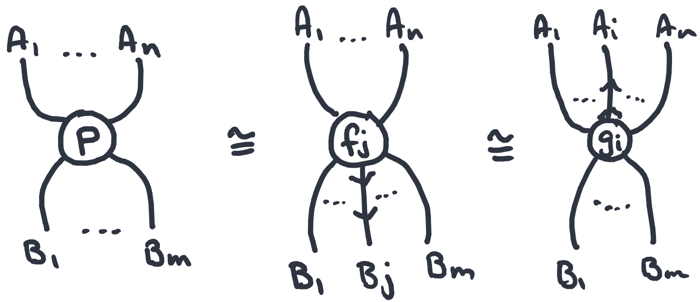


Here $P : A_1^{op} \times A_2^{op} \times B_1 \times B_2 \times B_3 \to SET$ is representable in B_1

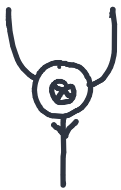
$$P(a_1, a_2, b_1, b_2, b_3) \cong$$
$$\hom(f_1(a_1, a_2, b_2, b_3), b_1).$$

Graphical Calculus.

DEF. A multivariable adjunction is an isomorphism clique.



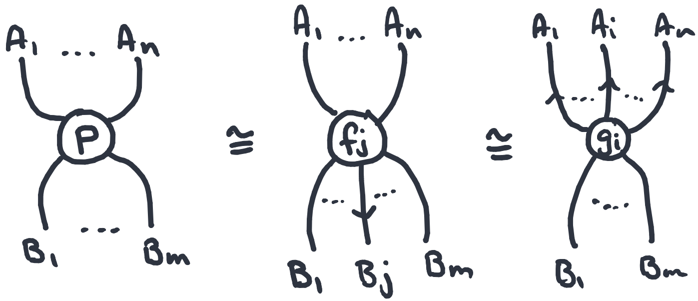
An example



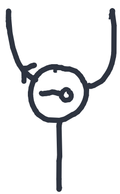
$\hom(A \otimes B, C)$

Graphical Calculus.

DEF. A multivariable adjunction is an isomorphism clique.



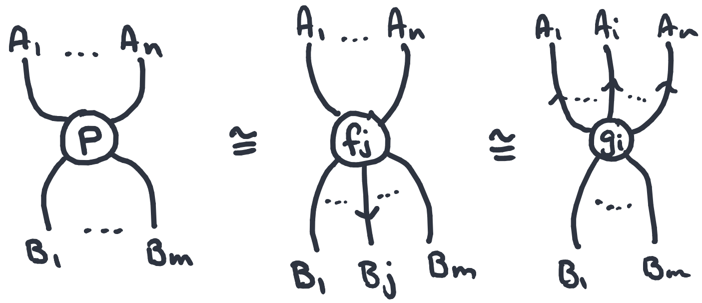
An example



$$\hom(B, A \multimap C)$$

Graphical Calculus.

DEF. A multivariable adjunction is an isomorphism clique.

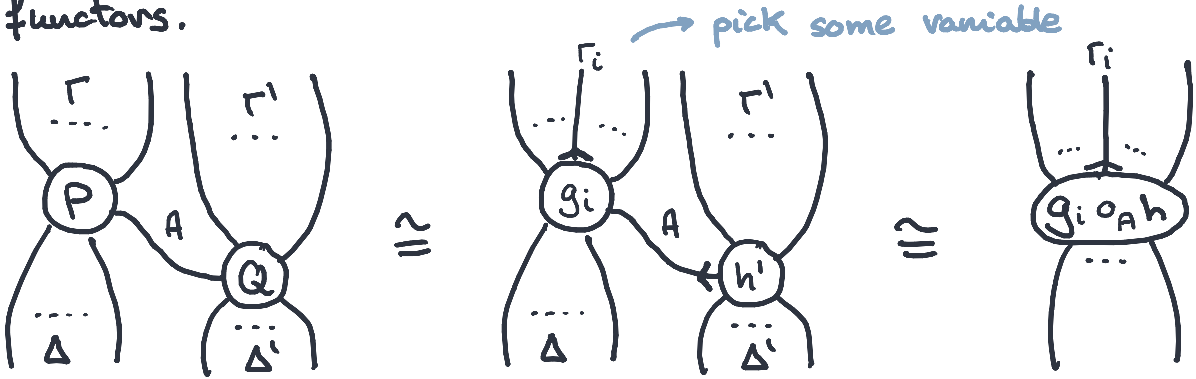


An example

$$\hom(A, C \multimap B)$$

Graphical Calculus.

Polycategorical composition of multivariable adjunctions gets defined by the multicategorical composition of the representable functors.



Compare with the previous proof.

$$\int^{a\in A} P(\Gamma_i, \Gamma_{\neq i}; a, \Delta) \times Q(a, \Gamma' ; \Delta') \cong$$

$$\int^{a\in A} \hom(\Gamma_i, g_i(\Gamma_{\neq i}, a, \Delta)) \times \hom(a, h'(\Gamma', \Delta')) \cong$$

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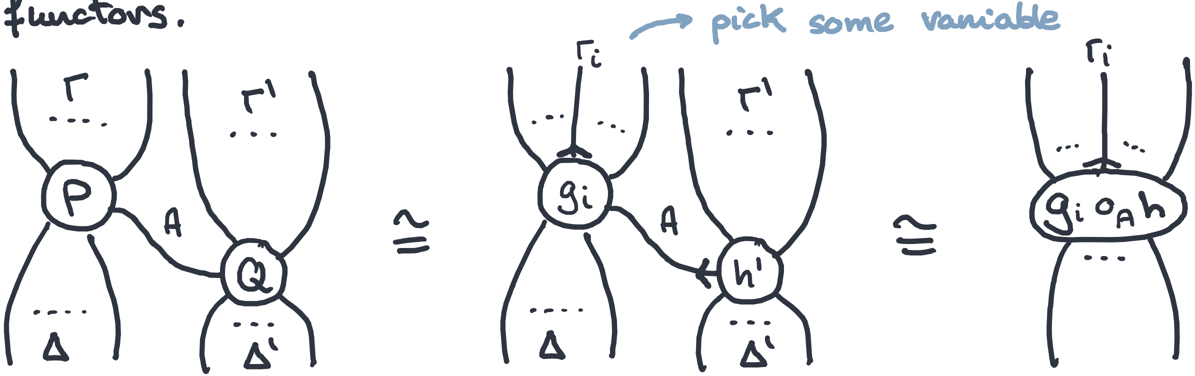
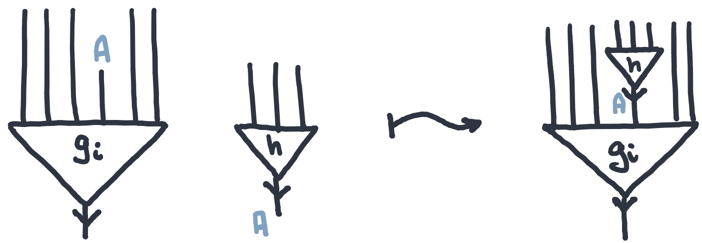
Graphical Calculus.

Polycategorical composition of multivariable adjunctions gets defined by the multicategorical composition of the representable functors.



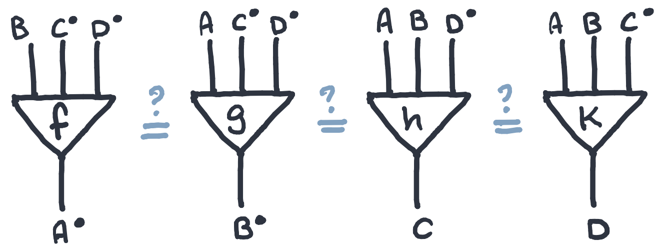
Polycategorical composition of these representables is enabled by multicategorical composition "locally".



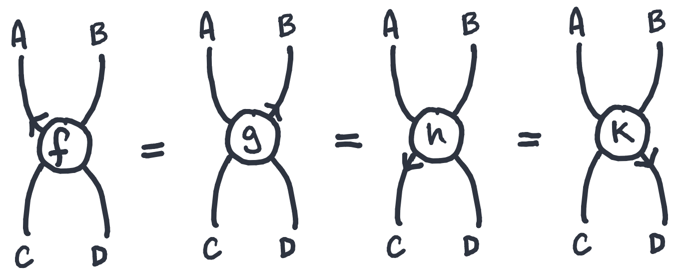
This is the Chu Construction.

with multiarrows $\Gamma \to ()$.

Given a multicategory, we want to construct a polycategory with duals. Morphisms should be cliques.



We would want to put equalities, but that does not typecheck.



"Pulling" in each direction we get the multiarrow.

This is the Chu Construction.

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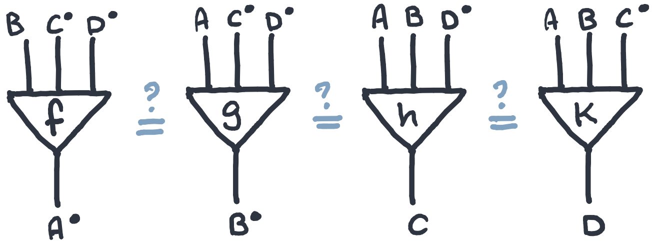

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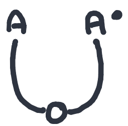
In a multicategory with empty codomains, we can ask each object to have a chosen "formal dual".



Objects

$(A, A^\circ; (A, A^\circ) \to ())$

This is the Chu Construction.

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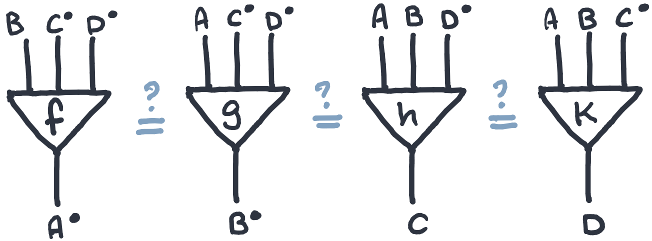

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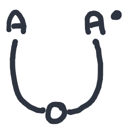
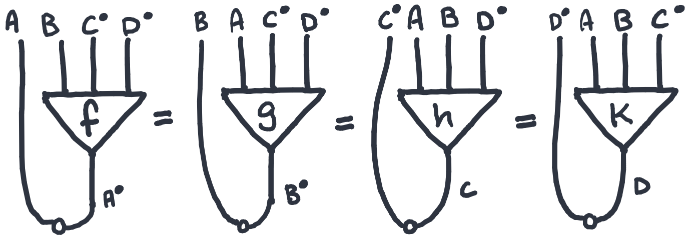


Object

$(A, A^\circ, (A, A^\circ) \to ())$

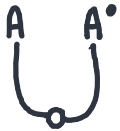
Now, this typechecks.



Morphisms.

Polycategorical Chu Construction.

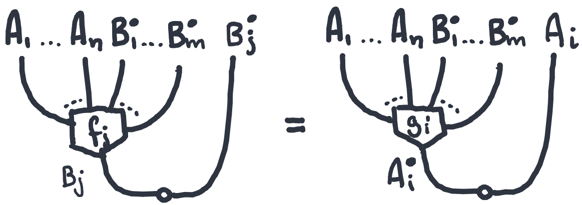
Objects.

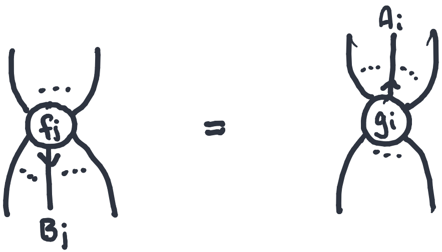


$(A, A^\circ, (A, A^\circ) \to ())$

Polyarrows are cliques.

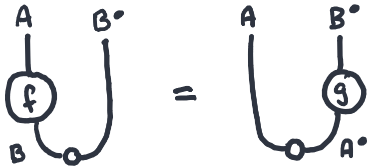
$(A_1 \dots A_n) \to (B_1 \dots B_m)$





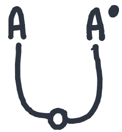
Usual Chu Construction

- Objects $(A, A^\circ, A \otimes A^\circ \to \bot)$
- Morphisms



Polycategorical Chu Construction.

Objects.



$(A, A^\circ, (A, A^\circ) \to ())$

Polyarrows $(A_1 \ldots A_n) \to (B_1 \ldots B_m)$
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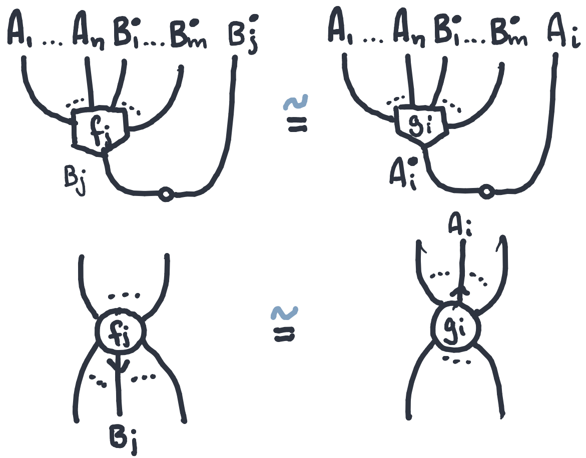
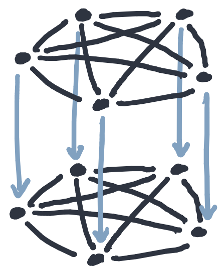


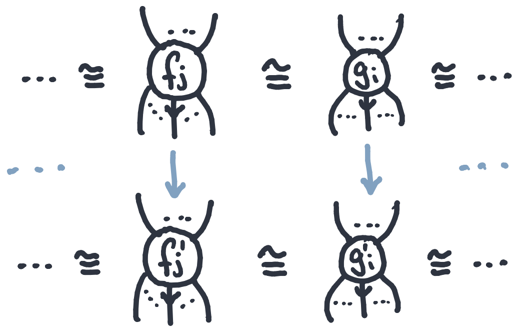
How to add the 2-cells?

A homomorphism of n-cliques is a family of maps.



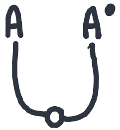
It is completely determined by its value at any component.

In our case,



Polycategorical Chu Construction.

Objects.



$(A, A^\circ, (A, A^\circ) \to ())$

Polyarrows $(A_1 \dots A_n) \to (B_1 \dots B_m)$
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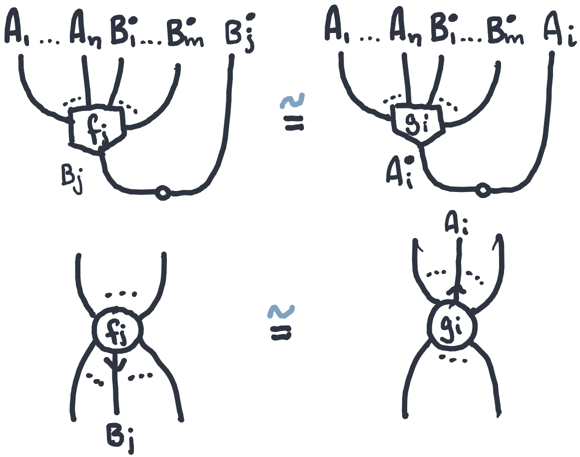


Multivariable adjunctions?

CAT is a multicategory where

$$A_1, \dots, A_n \to B_1$$

means $A_1 \times \dots \times A_n \to B_1$.
And we can take $A_1 \dots A_n \to ()$ to mean

$$A_1 \times \dots \times A_n \to Set.$$

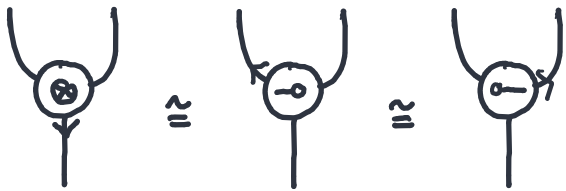
Multivariable adjunctions are a subbicategory of Chu(Cat, Set).

And this is true because of the universal property of Chu.

$$\begin{tikzcd} {*Poly} \arrow[r, bend left, "u"] \arrow[r, phantom, "\perp"] & Poly_{\leq 0} \arrow[l, bend left, "Chu"] \end{tikzcd}$$

Pseudomonoids in MADJ are closed categories.

The 1-cells give the functors of the closed category.



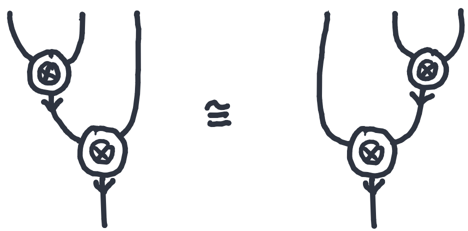
$\hom(A \otimes B, C) \cong \hom(B, A \multimap C) \cong \hom(A, C \multimapinv B)$

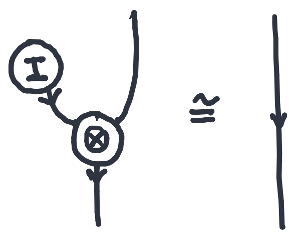
(2,1)-Adjunction

$\hom(I, A)$

(0,1)-Adjunction

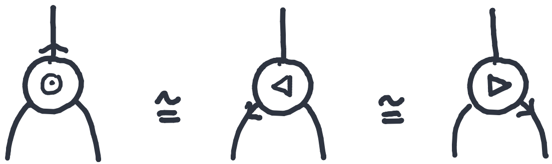
The 2-cells are completely determined by the component on $(\otimes)$.





Pseudo$^{\text{co!}}$monoids in MADJ are $^{\text{co!}}$closed categories.

The 1-cells give the functors of the $^{\text{co!}}$closed category.



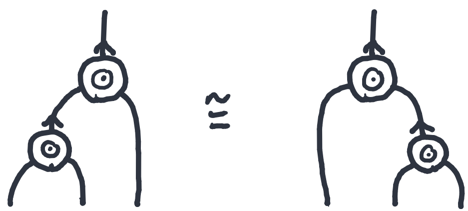
$\hom(A, B \odot C) \cong \hom(A \triangleleft B, C) \cong \hom(C \triangleright A, B)$

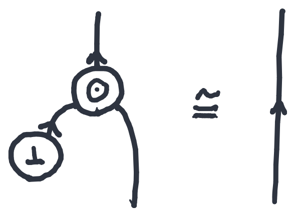
(1,2)-Adjunction

$\hom(A, \perp)$

(0,1)-Adjunction

The 2-cells are completely determined by the component on $(\otimes)$.

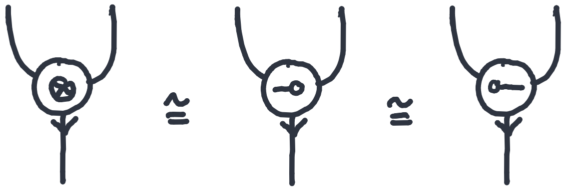


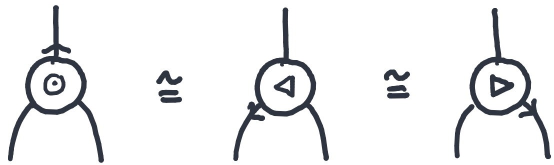


Linearly distributive categories

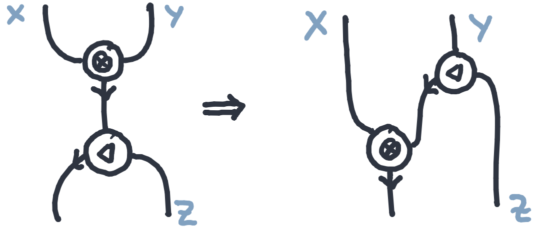
Linearly distributive categories which are closed and coclosed are lax Frobenius monoids in MADJ.

Sketch of the Proof. The monoid/comonoid pair gives $(\otimes, \odot)$.





The Frobenius rule gives,



$(X \otimes Y) \triangleleft Z \longrightarrow X \otimes (Y \triangleleft Z)$

We only need to get the linear distributors

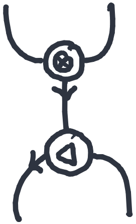
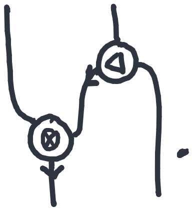
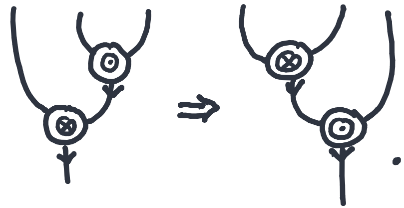
$$X \otimes (Y \odot Z) \longrightarrow (X \otimes Y) \odot Z$$

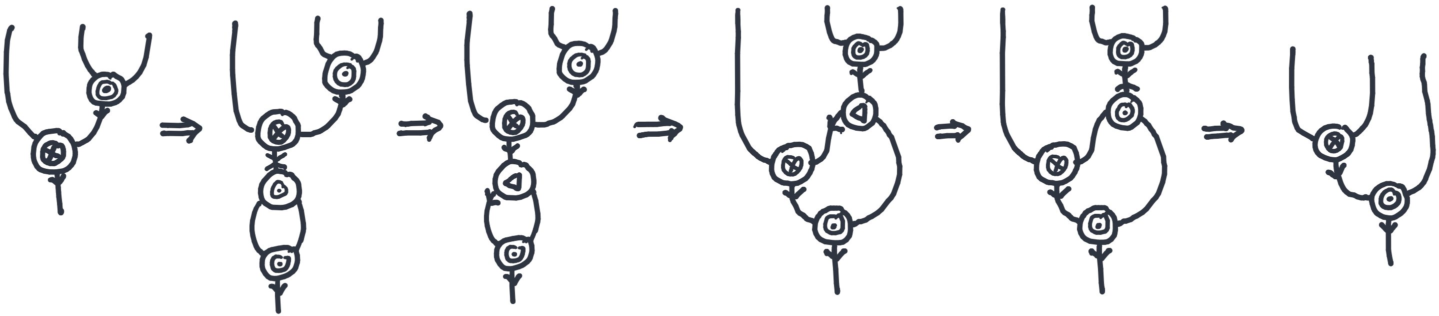
from

$$(X \otimes Y) \triangleleft Z \longrightarrow X \otimes (Y \triangleleft Z).$$

Linearly distributive categories

Linearly distributive categories which are closed and coclosed are lax Frobenius monoids in MADJ.

We have  $\xRightarrow{(frb)}$ . We want .



References.

- The 2-Chu-Dialectica construction. M. Shulman
- Star-Autonomous Categories are Frob. pseudomonoids. M. Shulman

If you want more diagrams:

- Open Diagrams Via Coend Calculus. M. R.

The 'mates' correspondence:

- Multivariable Adjunctions and Mates. Cheng, Riehl, Gurski.