Monoidal Categories

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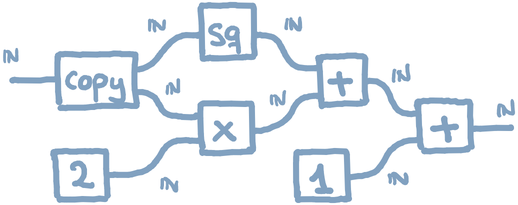
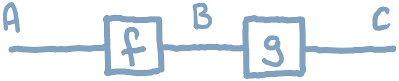


1-DIMENSIONAL DIAGRAMS

When writing/reading category theory, we denote $(f;g)$ as

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

By replacing lines by points and points by lines, we obtain a 1-dimensional diagram.



$$:= (f;g)$$

$$:= id_A$$

Composition is concatenation.

Categories have the axioms that make these diagrams preserve their value under different readings.

$$:= (f;g);h$$
$$= f;(g;h).$$

(associativity)

$$:= (f;id_B) = f = (id_A;f).$$

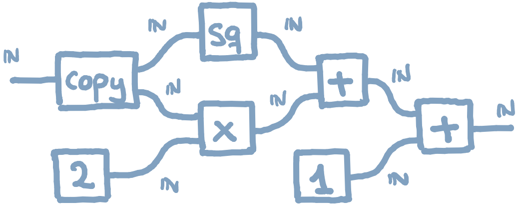
(unitality)

There is a unique morphism behind each diagram.

THE PROCESS INTERPRETATION

We interpret objects as *systems* or resources; we interpret morphisms as processes transforming these resources. Composing sequentially multiple processes, we get new ones.

However, we may want to consider multiple arieties and processes in parallel. We start using 2-dimensional string diagrams

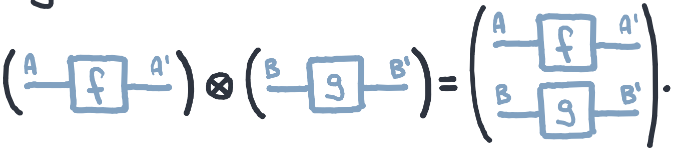


What are the data and the equations we need to make sense of these 2-dimensional string diagrams? Monoidal Categories.

Motivating Monoidal Categories

Definition. A monoidal category is a category $\mathbb{C}$ with, for each two objects, A and B, the tensor object $A \otimes B$,

and for each two morphisms $f : A \to A'$ and $g : B \to B'$, the tensor morphism $f \otimes g : A \otimes B \to A' \otimes B'$.



We ask $(\otimes) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ to be a functor.

It has a unit object $I \in \text{ob}\,\mathbb{C}$.

We need some natural isomorphisms.

$\alpha_{ABC} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$,

$\lambda_A : I \otimes A \to A$,

$\rho_A : A \otimes I \to A$.

They will be invisible under the graphical calculus.

Every formal equation between $\alpha, \lambda, \rho, \otimes, id$ must hold.

This is to say diagrams work.

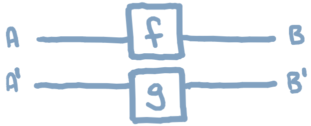
Ex: $\lambda_I = \rho_I$.

INTERCHANGE LAW

THEOREM (Interchange Law).
Let $(M, \otimes, I)$ be a monoidal category and let $f : A \to B$ and $g : A' \to B'$ be morphisms. The interchange law says:

$$(f \otimes id_{A'}) ; (id_B \otimes g) = (id_A \otimes g) ; (f \otimes id_{B'}).$$

The graphical calculus makes this law a deformation.



PROOF. We need to use that $(\otimes)$ is a functor.

$$\begin{aligned}
& (f \otimes id_A) ; (id_B \otimes g) \\
= \; & \qquad\qquad\qquad \text{notation} \\
& \otimes(f, id_A) ; \otimes(id_B, g) \\
= \; & \qquad\qquad\qquad \otimes \text{ is a functor} \\
& \otimes((f, id_A) ; (id_B, g)) \\
= \; & \qquad\qquad\qquad \text{composition in } \mathbb{C} \times \mathbb{C} \\
& \otimes((f ; id_B), (id_A ; g)) \\
= \; & \qquad\qquad\qquad \text{identities are neutral} \\
& \otimes(f, g) \\
= \; & \qquad\qquad\qquad \text{notation} \\
& f \otimes g
\end{aligned}$$

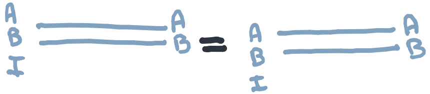
COHERENCE FOR MONOIDAL CATEGORIES.

EXAMPLE. The following equation holds

$$\begin{tikzcd}
(A\otimes B)\otimes I \arrow[r, "\rho_{A\otimes B}"] \arrow[d, "\alpha_{ABI}"'] & A\otimes B \\
A\otimes(B\otimes I) \arrow[ur, bend right, "id_A\otimes\rho_B"] &
\end{tikzcd}$$

$$\alpha_{ABI} \,;\, (id_A \otimes \rho_B) = \rho_{A\otimes B}$$

Diagrams make the equation invisible.



REMARK. Proving that something is a monoidal category would be very difficult: we need to show that ALL diagrams commute. The COHERENCE THEOREM makes it easier.

THEOREM (Coherence). If the following "pentagons" and "triangles" commute, then all formal equations made up of $\alpha, \lambda, \rho, id, \otimes$ hold.

$$\begin{tikzcd}
((A\otimes B)\otimes C)\otimes D \arrow[r, "\alpha"] \arrow[d, "\alpha\otimes id"'] & (A\otimes B)\otimes(C\otimes D) \arrow[d, "\alpha"] \\
(A\otimes(B\otimes C))\otimes D \arrow[dr, "\alpha"'] & A\otimes(B\otimes(C\otimes D)) \arrow[l, phantom, "="] \\
& A\otimes((B\otimes C)\otimes D) \arrow[u, "id\otimes\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A\otimes I)\otimes B \arrow[dr, "\rho\otimes id"] \arrow[dd, "\alpha"'] & \\
= & A\otimes B \\
A\otimes(I\otimes B) \arrow[ur, "id\otimes\lambda"'] &
\end{tikzcd}$$

SETS IS A MONOIDAL CATEGORY

PROPOSITION. The category of sets has a monoidal structure $(\mathrm{SETS}, \times, 1)$ with the cartesian product and the singleton set.

Proof. We have already defined the cartesian product $A \times B = \{(a,b) \mid a \in A, b \in B\}$. We can also define the product of morphisms $(f \times g) : A \times B \to A' \times B'$ as $(f \times g)(a,b) = (fa, gb)$. This is functorial, and we can check that via elements.

$$(id \times id)(a,b) = (a,b)$$

$$(f' \times g')(f \times g)(a,b) = (f'fa, g'gb) = (f'f \times g'g)(a,b)$$

We take the unit to be $1 = \{*\}$ the singleton set.

Now, we can define the natural isomorphisms.

$$\lambda(*,a) = a \qquad \rho(a,*) = a \qquad \alpha((a,b),c) = (a,(b,c)).$$

And finally use the coherence theorem to just check the triangle and pentagon equations.

$$(id \times \lambda)\,\alpha((a,*),b) = (id \times \lambda)(a,(*,b)) = (a,b)$$
$$= (\rho \times id((a,*),b).$$

$$\alpha\alpha(((a,b),c),d) = \alpha((a,b),(c,d))$$
$$= (a,(b,(c,d))) = (id \times \alpha)(a,((b,c),d))$$
$$= (id \times \alpha)\alpha((a,(b,c)),d)$$
$$= (id \times \alpha)\alpha(\alpha \times id)\ (((a,b),c),d).$$

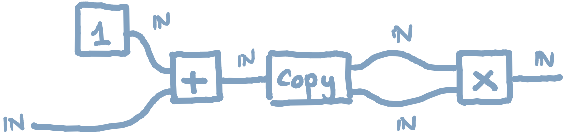
Exercise. Set is also a monoidal category with the disjoint union of sets and the empty set $(\mathrm{SET}, +, 0)$.

EXAMPLES OF MONOIDAL CATEGORY

EXAMPLE. The category of sets and functions $(\mathbf{SET}, \times, 1)$, using the product of $f : A \to B$ and $g : A' \to B'$ as tensor $(f \times g) : A \times A' \to B \times B'$.

$$(f \times g)(a, a') = (fa, ga').$$

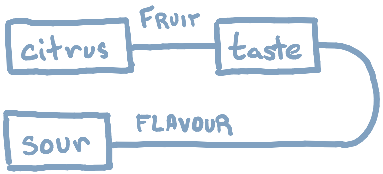
The unit is the singleton set.



EXAMPLE. The category of sets and relations $(\mathbf{REL}, \times, 1)$, taking relations $R : A \to B$ and $S : A' \to B'$ to the relation $R \times S : A \times A' \to B \times B'$ given by

$$(a, a')\, R{\times}S\, (b, b') = (aRb) \wedge (a'Sb')$$

The unit is again the singleton set.



Definitions:

citrus:	taste:	sour:
(•, lemon)	(lemon, sour)	(sour, •)
(•, orange)	(orange, sour)	
	(orange, sweet)	

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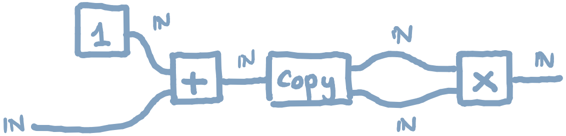

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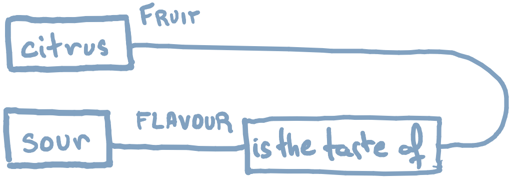

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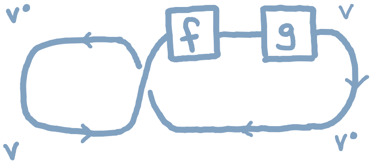
EXAMPLES OF MONOIDAL CATEGORY

EXAMPLE. The category of vector spaces and linear functions $(\mathrm{VECT}, \otimes, \mathbb{R})$, taking $(\varphi \otimes \psi) : A \otimes B \to A' \otimes B'$ to be defined by

$$(\varphi \otimes \psi)\left(\sum_{ij} \lambda_{ij} (a_i \otimes b_j)\right) = \sum_{ij} \lambda_{ij} (\varphi a_i \otimes \psi b_j).$$

The unit is $\mathbb{R}$; the same works for R-modules for an arbitrary ring R.



$= tr(f;g)$

EXAMPLE. The category of endofunctors of any category and natural transformations $([\mathbb{C},\mathbb{C}], \circ, \mathrm{Id})$ with $\alpha * \beta : F \circ F' \to G \circ G'$ defined by

$$\begin{tikzcd}
FF'X \arrow[r, "\alpha_{F'X}"] \arrow[d, "F\beta_X"'] \arrow[dr, "(\alpha*\beta)_X"] & GF'X \arrow[d, "G\beta_X"] \\
FG'X \arrow[r, "\alpha_{G'X}"'] & GG'X
\end{tikzcd}$$

.

The unit is the identity functor ID. This category is strict monoidal.

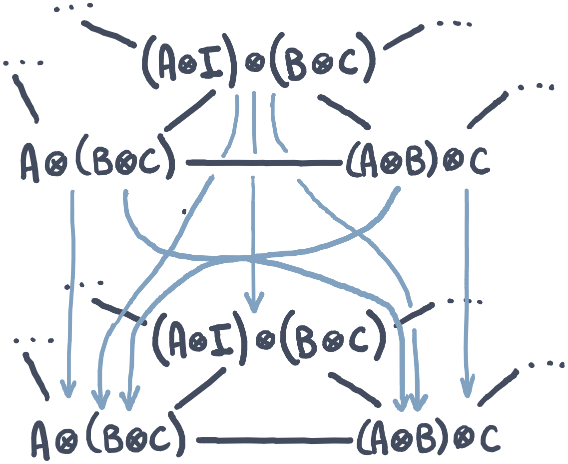
STRICT MONOIDAL CATEGORIES

DEFINITION. A monoidal category is **strict** when associators and unitors are identity natural transformations. This makes

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C,$$
$$A \otimes I = A = I \otimes A.$$

THEOREM (Strictification). Every monoidal category is equivalent to a strict one.

PROOF. We embed a monoidal $\mathcal{M}$ into a strict monoidal category $\mathcal{M}^{st}$. The category $\mathcal{M}^{st}$ has cliques of coherence isomorphisms as objects and families of morphisms between them as morphisms; making each diagram commute.



All diagrams commute.

This category is strict, with

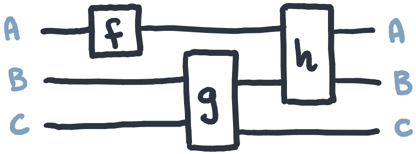
$$\alpha = id, \quad \lambda = id, \quad \rho = id.$$

The inclusion $i : \mathcal{M} \longrightarrow \mathcal{M}^{st}$ also preserves tensors.

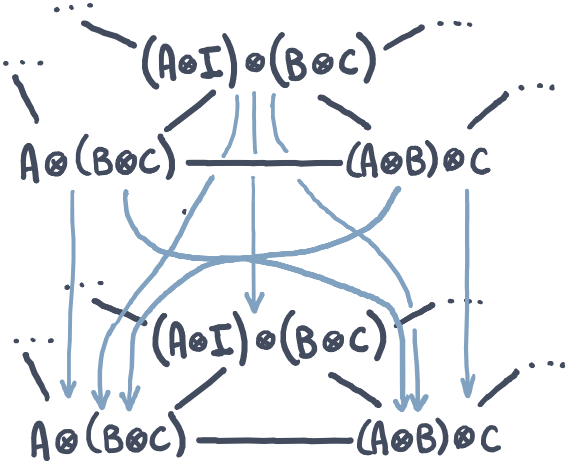
$$i(A) \otimes i(B) = i(A \otimes B).$$

STRICT MONOIDAL CATEGORIES (AND DIAGRAMS)

String diagrams alone determine a morphism only on a strict category.



$$((f \otimes id) \otimes id) ; \alpha ; (id \otimes g) ; \alpha^{-1} ; (h \otimes id)$$
$$: (A \otimes B) \otimes C \longrightarrow (A \otimes B) \otimes C$$



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We can draw diagrams for the strictification of a monoidal category, and then pick the domain and the codomain, determining the morphism.

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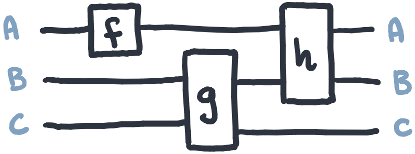
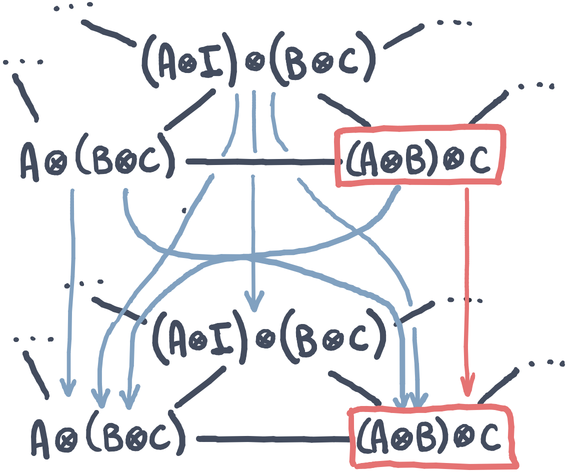


$$((f \otimes id) \otimes id) ; \alpha ; (id \otimes g) ; \alpha^{-1} ; (h \otimes id)$$
$$: (A \otimes B) \otimes C \longrightarrow (A \otimes B) \otimes C$$



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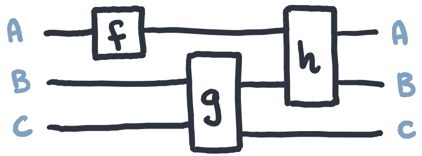
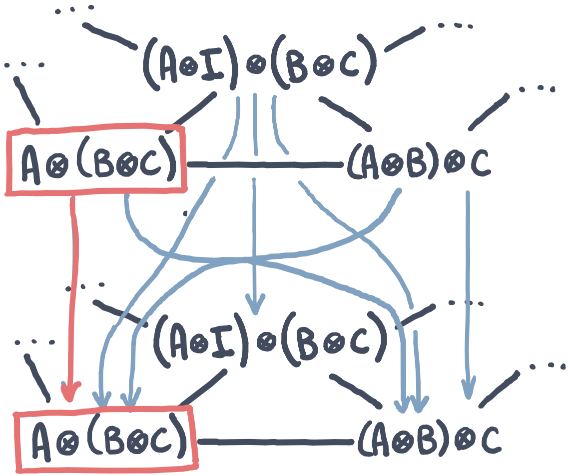


$$(f\otimes id);(id\otimes g);\alpha^{-1};(h\otimes id);\alpha$$
$$: A\otimes(B\otimes C) \longrightarrow A\otimes(B\otimes C)$$



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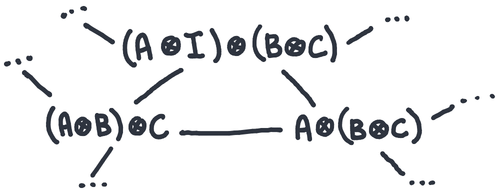
We can draw diagrams for the strictification of a monoidal category, and then pick the domain and the codomain, determining the morphism.

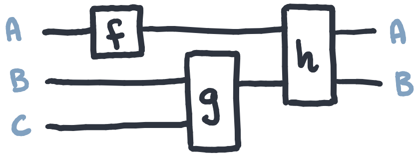
STRICTIFICATION

Let M be a monoidal category. Its strictification, M^{st} has

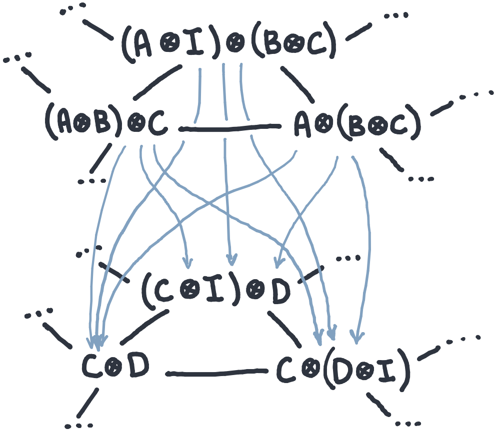
- Objects cliques of coherence isomorphisms



they commute because of the definition.



- Morphisms are families of morphisms commuting with coherence cliques.

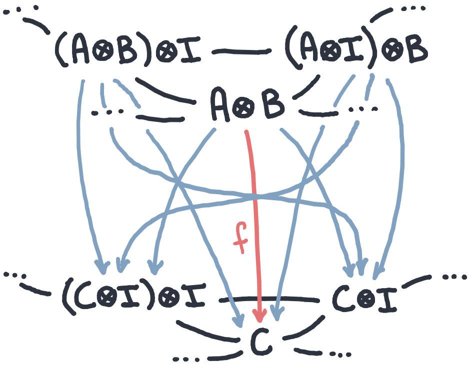


Any given component determines all the others by isomorphism.

STRICTIFICATION

We have an embedding $M \longrightarrow M^{st}$.

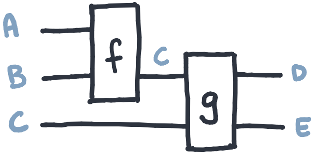
$$\begin{tikzcd} A \otimes B \arrow[d, "f"'] \\ C \end{tikzcd}$$



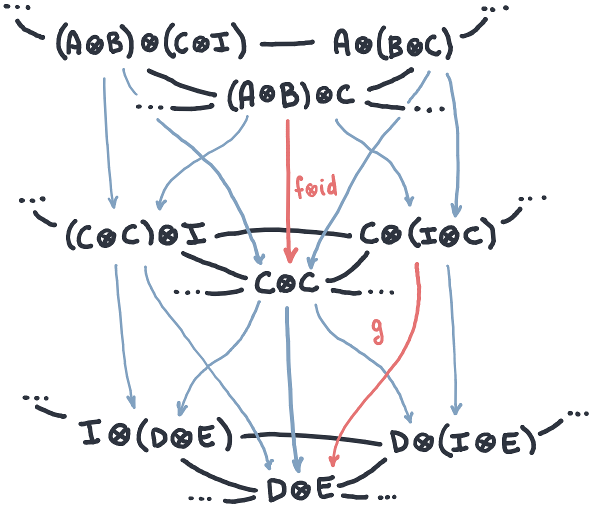
The embedding preserves tensors.

$$e(A\otimes B) = e(A)\otimes e(B).$$

We use the embedding to read diagrams. for $f : A\otimes B \to C$, $g : C\otimes(I\otimes C) \to D\otimes E$.



Can be interpreted with different domains.



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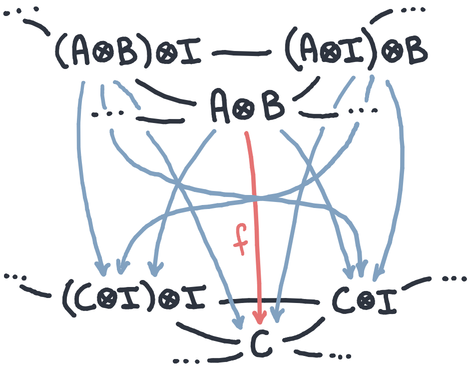

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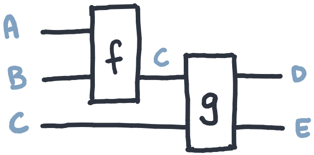
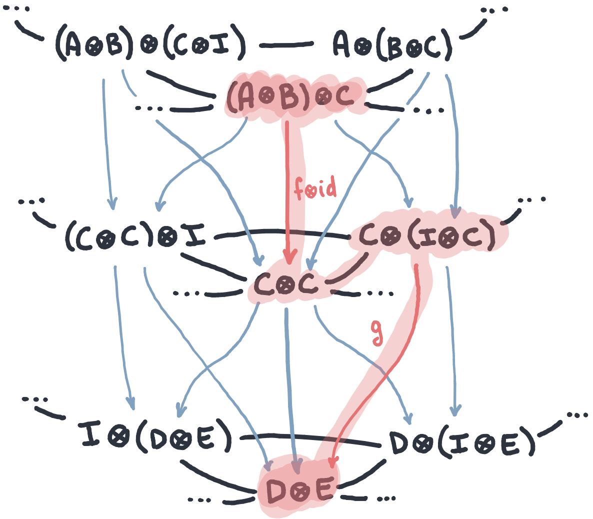


Can be interpreted as $(A \otimes B) \otimes C \to D \otimes E$



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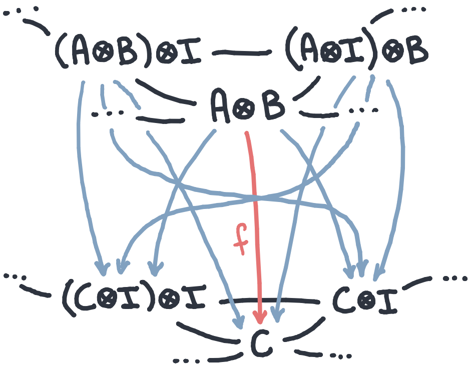

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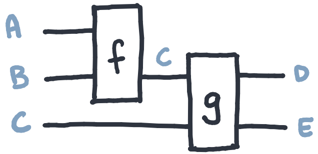

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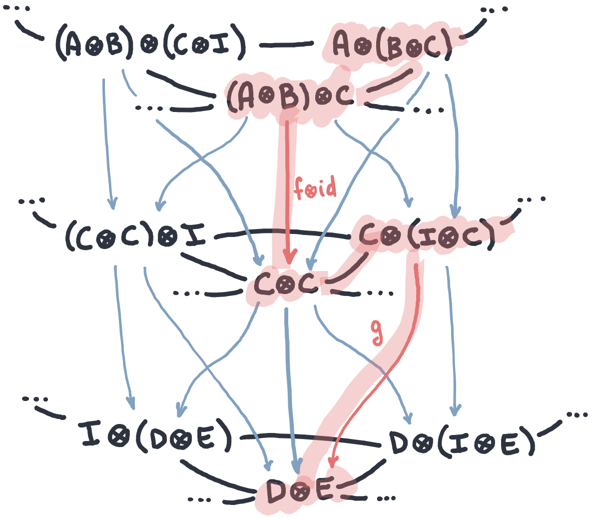

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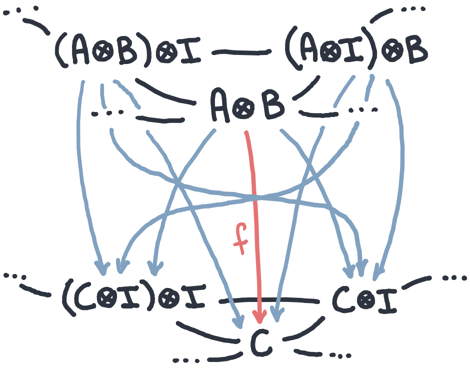

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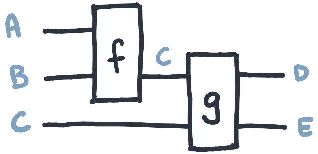

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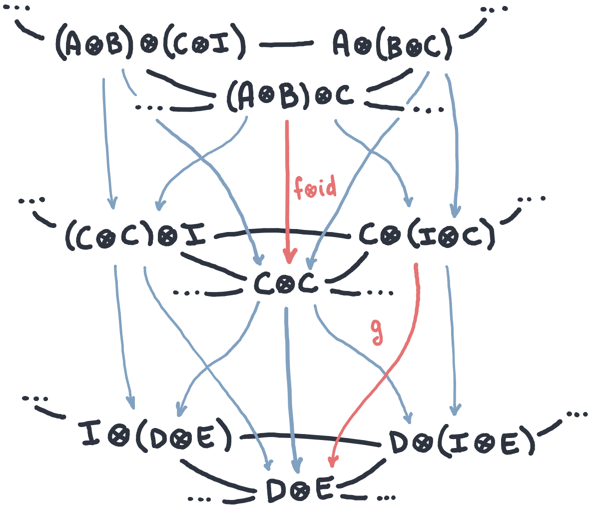
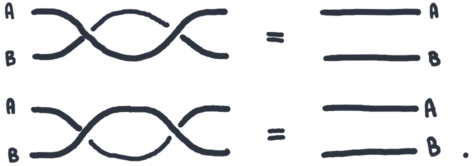


BRAIDED MONOIDAL CATEGORIES

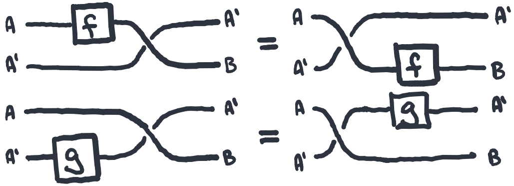
DEFINITION. A braided monoidal category is a monoidal category endowed with a natural isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$, called the "braiding", that satisfies the hexagon equations.

We denote $\sigma_{A,B}$ by  and $\sigma^{-1}_{A,B}$ by .

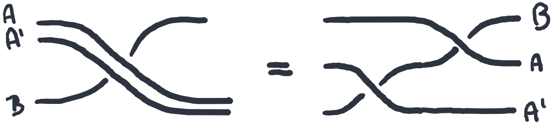
ISOMORPHISM:



NATURAL:



HEXAGON EQUATIONS:



$$\sigma_{A\otimes A',B} = \alpha_{AA'B} ; (id_A \otimes \sigma_{A',B}) ; \alpha^{-1}_{ABA'} ; (\sigma_{A,B} \otimes id_{A'}) ; \alpha_{BAA'}$$

$$\sigma_{A,B\otimes B'} = (\sigma_{A,B} \otimes id_{B'}) ; (id_B \otimes \sigma_{A,B'}) .$$

DEFINITION. A symmetric monoidal category is a braided monoidal category with $\sigma_{A,B} ; \sigma_{B,A} = id_{A\otimes B}$.

We write $\sigma_{AB} :=$ 

BRAIDED MONOIDAL CATEGORIES: EXAMPLES

In (SET, ×, 1) there exists a function $\sigma : A\times B \to B\times A$ defined as $\sigma(a,b) = (b,a)$. It has an inverse $\sigma^{-1}(b,a) = (a,b)$; and it is natural

$$\sigma(fa,b) = (b,fa) = (\mathrm{id}\times f)\sigma(a,b)$$

$$\sigma(a,gb) = (gb,a) = (g\times\mathrm{id})\sigma(a,b)$$

$\shortparallel$ $\shortparallel$

$\sigma(\mathrm{id}\times g)(a,b)$ $(g\times\mathrm{id})(b,a)$

HEXAGON EQUATION(S).

$$\sigma_{A\times A',B}((a,a'),b)$$
$$= (b,(a,a'))$$
$$\alpha(\sigma\times\mathrm{id})\alpha^{-1}(\mathrm{id}\times\sigma)\alpha((a,a'),b)$$
$$= \alpha(\sigma\times\mathrm{id})\alpha^{-1}(\mathrm{id}\times\sigma)(a,(a',b))$$
$$= \alpha(\sigma\times\mathrm{id})\alpha^{-1}(a,(b,a'))$$
$$= \alpha(\sigma\times\mathrm{id})((a,b),a')$$
$$= \alpha((b,a),a')$$
$$= (b,(a,a'))$$

EXAMPLE (Yang-Baxter equation).

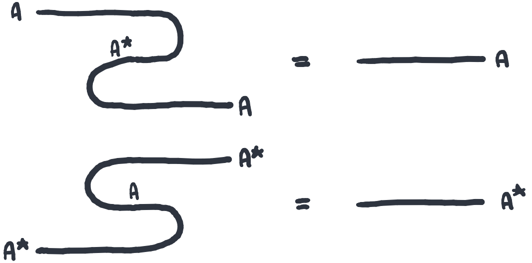
DUALS AND COMPACT CLOSED CATEGORIES

DEFINITION. An object $A \in \text{ob}\,\mathcal{M}$ is said to have a **dual** $A^* \in \text{ob}(\mathcal{M})$ when there are maps

$\eta : I \to A \otimes A^*$, called unit or **cup**;

$\varepsilon : A^* \otimes A \longrightarrow I$, called counit or **cap**;

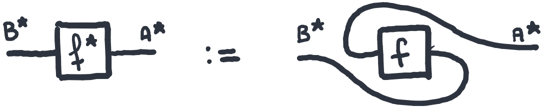
satisfying the **snake equations**.



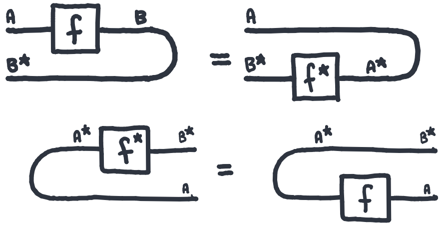
DEFINITION. A symmetric monoidal category $\mathcal{M}$ is **compact closed** if every object is dualizable.

PROPOSITION. In a symmetric monoidal category, the dual of the dual is isomorphic to the original, $A^{**} \cong A$.

DEFINITION. Every morphism $f : A \to B$ induces a dual morphism $f^* : B^* \to A^*$.



PROPOSITION. The following equations hold.



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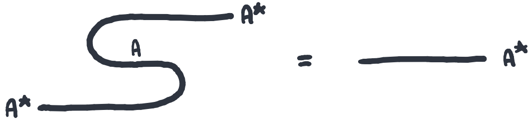

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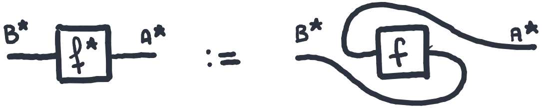


EXAMPLE. In the category of sets and relations we have a cup and cap.

$:= \{ (a,a) E * \mid a \in A \}$

$:= \{ * E (a,a) \mid a \in A \}$

MONOIDS AND COMONOIDS

Classically, a monoid is a set M together with functions $m: M \times M \to M$ and $e: 1 \to M$. This definition can be repeated in any monoidal category with tensors and units. Rewriting also its axiom

$$m(m(a,b),c) = m(a,m(b,c)) \qquad m(e,a = m(a,e) = a$$

DEFINITION. A monoid in a monoidal category $(\mathbb{C}, \otimes, I)$ is an object M together with morphisms

$$m: M \otimes M \to M \quad \text{and} \quad e: I \to M,$$

such that the following axioms hold.

This definition can be interpreted in any monoidal category, but it can be also dualized.

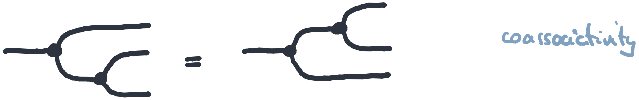
DEFINITION. A comonoid in $\mathbb{C}$ is a monoid in $\mathbb{C}^{op}$. We have morphisms

$$c: A \to A \otimes A, \qquad d: A \to I,$$

$c =$ 

$d =$ 

such that the following axioms hold.



EXAMPLES.

- $(\text{SET}, \times, 1)$: usual monoids.
- $(\text{MON}, \times, 1)$: abelian monoid
- $(\text{TOP}, \times, \{*\})$: topological monoid.
- $([\mathbb{C},\mathbb{C}], \circ, \text{Id})$: monad.

$$e: \text{Id} \Rightarrow T, \qquad e_X : X \to TX$$

$$m: T \circ T \Rightarrow T, \qquad m_X : TTX \to TX$$

$$\begin{tikzcd}
TTX \arrow[dr, "m"'] & TX \arrow[l, "Te_X"'] \arrow[r, "e_{TX}"] \arrow[d, no head] \arrow[dl, phantom, "="] \arrow[dr, phantom, "="] & TTX \arrow[dl, "m"] \\
 & TX &
\end{tikzcd}
\qquad
\begin{tikzcd}
TTTX \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"'] \arrow[dr, phantom, "//"] & TTX \arrow[d, "\mu_X"] \\
TTX \arrow[r, "\mu_X"] & TX
\end{tikzcd}$$

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DEFINITION. A monoid in the category $(\mathrm{SET}, \times, 1)$ is a set M together with functions

$$\cdot : M \times M \to M \quad \text{and} \quad e : 1 \to M,$$

such that the following axioms hold.

unitality

$a \cdot e = a = e \cdot a$

associativity

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

This definition can be interpreted in any monoidal category, but it can be also dualized.

DEFINITION. A comonoid in $\mathbb{C}$ is a monoid in $\mathbb{C}^{op}$. We have morphisms

$$c : A \to A \otimes A, \qquad d : A \to I,$$

such that the following axioms hold.

counitality

coassociativity

EXAMPLES.

- $(\mathrm{SET}, \times, 1)$: usual monoids.
- $(\mathrm{MON}, \times, 1)$: abelian monoid ↖ Eckmann-Hilton.
- $(\mathrm{TOP}, \times, \{*\})$: topological monoid.
- $([\mathbb{C}, \mathbb{C}], \circ, \mathrm{Id})$: monad.

$$e : \mathrm{Id} \Rightarrow T, \qquad e_x : X \to TX$$

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TTX \arrow[dr, "m"'] & TX \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, no head] & TTX \arrow[dl, "m"] \\
& TX &
\end{tikzcd}
\qquad
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TTX \arrow[r, "\mu_x"] & TX
\end{tikzcd}$$

CARTESIAN CATEGORIES ARE MONOIDAL

Reasoning in SET is easier because we can compute with elements. Could we do this in other categories? We will use generalized elements and the Yoneda lemma.

THEOREM. Cartesian categories are monoidal.

Proof. We write $(a,b) : X \to A\times B$ for the unique morphism with projections $a : X \to A$ and $b : X \to B$. In order to define the tensor $(f\times g)$ we define a function

$$\overline{(f\times g)} : \hom(Z, A\times B) \to \hom(Z, A'\times B')$$
$$\overline{(f\times g)}(a,b) = (fa, gb).$$

$a : Z \to A$!
$b : Z \to B$!

This function is natural on Z. Let $h : Z' \to Z$,

$$\overline{(f\times g)}((a,b)\circ h) = \overline{(f\times g)}(ah, bh) = (fah, gbh) =$$
$$= (fa, gb)h = (\overline{(f\times g)}(a,b))\circ h.$$

Thus, by Yoneda, we have a morphism

$$(f\times g) : A\times B \longrightarrow A'\times B'.$$

We could have simply defined it as $(f\pi_1, g\pi_2)$, but writing $(f\times g)(a,b) = (fa, gb)$ feels intuitive.

Let us show that $(\times)$ is functourial as in SETS:

$$(id\times id)(a,b) = (a,b)$$
$$(f'\times g')(f\times g)(a,b) = (f'fa, g'gb) = (f'f\times g'g)(a,b).$$

Using generalized elements, we can also define unitors and associators.

$$\lambda(*,a) = a, \text{ natural } \lambda((*,a)h) = \lambda(*h, ah) = \lambda(*, ah) = ah$$
$$\rho(a,*) = a, \text{ natural } \rho((a,*)h) = \rho(ah, *h) = \rho(ah, *) = ah$$
$$\alpha((a,b),c) = (a,(b,c)), \text{ natural}$$
$$\alpha(((a,b),c)h) = (ah,(bh,ch)) = (a,(b,c))h.$$

We check coherence conditions as in SET.

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This function is natural on Z. Let $h\colon Z'\to Z$,

$$(\overline{f\times g})((a,b)\circ h) = (f\times g)(ah, bh) = (fah, gbh) =$$
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Let us show that $(\times)$ is functorial as in SETS:

$$(id\times id)(a,b) = (a,b)$$
$$(f'\times g')(f\times g)(a,b) = (f'fa, g'gb) = (f'f\times g'g)(a,b).$$

$$(id\times\rho)\,\alpha((a,*),b) = (id\times\lambda)(a,(*,b)) = (a,b)$$
$$= \rho((a,*),b).$$

$$\alpha\alpha(((a,b),c),d) = \alpha((a,b),(c,d))$$
$$= (a,(b,(c,d))) = (id\times\alpha)(a,((b,c),d))$$
$$= (id\times\alpha)\alpha((a,(b,c)),d)$$
$$= (id\times\alpha)\alpha\alpha(((a,b),c),d).$$

COMONOIDS AND CARTESIAN CATEGORIES

Why aren't comonoids better known as algebraic structures? Why do we need to abstract monoids to be able to describe them? They appear only trivially in categories such as SET.

DEFINITION. A monoidal category is cartesian when the tensor product (×) and the unit coincides with the terminal object (1). Unitors and associators are given by the projections.

EXAMPLE. (SET, ×, 1) is cartesian monoidal.

THEOREM. In a cartesian category, each object is a comonoid in a unique way.

Proof. First note that the counit must be $\varepsilon = * : A \to 1$, the unique morphism to the terminal object. The diagonal must satisfy

$$\lambda(id \times \varepsilon)\delta = id = \rho(id \times \varepsilon)\delta$$

.

As we know that λ, ρ are projections, we have that the two projections of δ are the identities, $\delta = (id, id)$.

This only possibility is in fact a comonoid, we only need coassociativity.

$$(\delta \times id)\delta(a) = (\delta \times id)(a,a) = ((a,a),a),$$

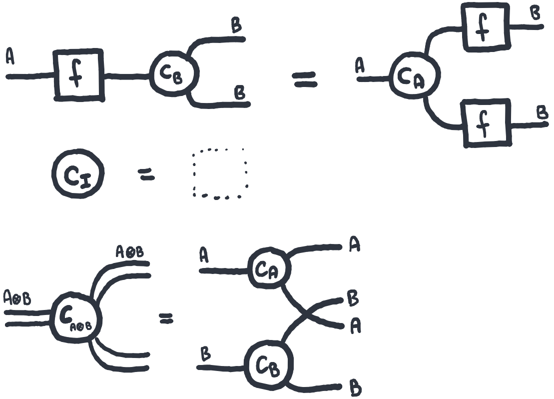
$$\alpha(id \times \delta)\delta(a) = \alpha((a,a),a) = ((a,a),a).$$

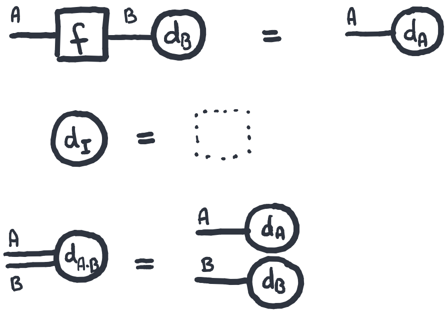
UNIFORM COPY-DELETE

DEFINITION. A monoidal category has uniform copy-delete when there exist families of morphisms 

and 

for each A, satisfying:





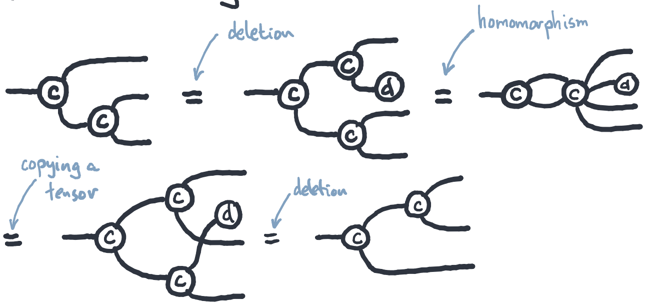
And additionally interacting as

.

FOX'S THEOREM

THEOREM. If a category has uniform copy-delete, then every object is a comonoid and every morphism is a comonoid homomorphism.

PROOF. We only need to prove (c_A, d_A) is a comonoid structure; and the only axiom missing is coassociativity



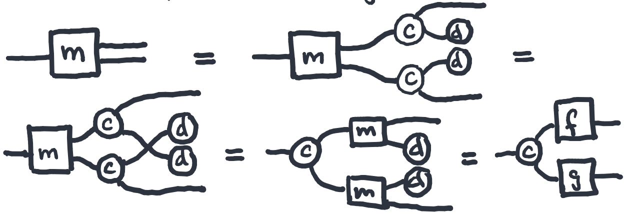
□

THEOREM. Every category with finite products has uniform copy-delete; but also any category with uniform copy-delete has finite products.

Proof. Let us show that I is a terminal object.

Let us show that $A \otimes B$ is a cartesian product. We define projections as

Given $f: C \to A$ and $g: C \to B$, there exists a unique morphism that projects into them.



□

COMONOIDS: EXAMPLES

A comonoid in $[\mathbb{C},\mathbb{C}]$ is a comonad.
Comonads are endofunctors $R : \mathbb{C} \to \mathbb{C}$ with natural transformations

$$\varepsilon : RX \longrightarrow X$$
$$\delta : RX \longrightarrow RRX$$

such that the following diagrams commute.

$$\begin{tikzcd}
RRX \arrow[dr, "\varepsilon_{RX}"'] & RX \arrow[l, "\delta_X"'] \arrow[r, "\delta_X"] \arrow[d, no head, "="', "="] & RRX \arrow[dl, "R\varepsilon_X"] \\
 & RX &
\end{tikzcd}
\qquad
\begin{tikzcd}
RX \arrow[r, "\delta_X"] \arrow[d, "\delta_X"'] \arrow[dr, phantom, "="] & RRX \arrow[d, "R\delta_X"] \\
RRX \arrow[r, "\delta_{RX}"'] & RRRX
\end{tikzcd}$$

EXAMPLE (Stream comonad).
Let $SX = \{(x_0, x_1, \dots) \mid x_n \in X\}$ be the set of infinite streams with entries on X.

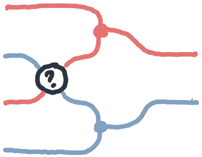
$$\varepsilon(x_0, x_1, x_2, \dots) = x_0$$
$$\delta(x_0, x_1, x_2, \dots) = ((x_0, x_1, x_2, \dots), (x_1, x_2, x_3, \dots), (x_2, x_3, x_4, \dots), \dots)$$

Check this gives a comonad.

String Diagrams for Distributive Laws

The composition of two monads is not a monad, why? This is a particular case of a more general phenomenon: the tensor of two monoids S and T is not in general a monoid.

The unit of $S \otimes T$ is clearly , but what is the multiplication?



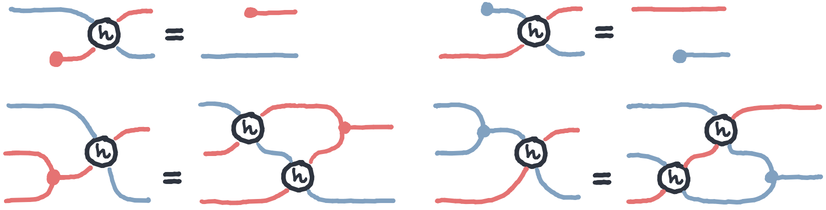
We would need some form of switching.

It would need some compatibility rules to satisfy the axioms.

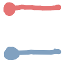
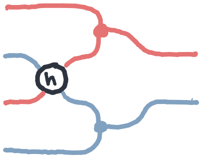
DEFINITION. A distributive law of a monoid S over a monoid T is a morphism

$$h : T \otimes S \longrightarrow S \otimes T$$

such that



THEOREM. Given two monoids S and T with a distributive law $h : T \otimes S \to S \otimes T$, we can give a monoid structure on $S \otimes T$.

Unit:  Multiplication: 

String Diagrams for Distributive Laws

DEFINITION. A distributive law of a monad (S,μ,η) over a monad (T,μ,η) is a natural transformation $h_A : TSA \to STA$ such that the following diagrams commute.

$$\begin{tikzcd}
TA \arrow[r, "T\eta_A"] \arrow[d, "\eta_{TA}"'] & TSA \arrow[dl, "h_A"'] \\
STA &
\end{tikzcd}
\qquad
\begin{tikzcd}
SA \arrow[r, "\eta_{SA}"] \arrow[d, "S\eta_A"'] & TSA \arrow[dl, "h_A"'] \\
STA &
\end{tikzcd}$$

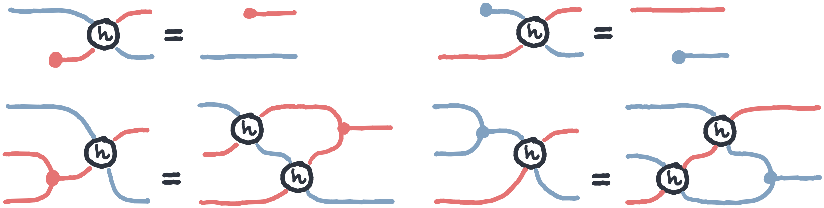
$$\begin{tikzcd}
TSSA \arrow[r, "h_{SA}"] \arrow[d, "T\mu_A"'] & STSA \arrow[r, "Sh_A"] & SSTA \arrow[d, "\mu_{TA}"] \\
TSA \arrow[rr, "h_A"] & & STA
\end{tikzcd}$$

$$\begin{tikzcd}
TTSA \arrow[r, "Th_A"] \arrow[d, "\mu_{SA}"'] & TSTA \arrow[r, "h_{TA}"] & STTA \arrow[d, "\mu_{SA}"] \\
TSA \arrow[rr, "h_A"] & & STA
\end{tikzcd}$$

DEFINITION. A distributive law of a monoid S over a monoid T is a morphism

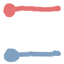
$h : T \otimes S \longrightarrow S \otimes T$

such that

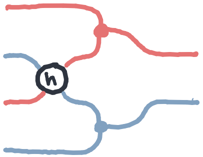


THEOREM. Given two monoids S and T with a distributive law $h : T \otimes S \to S \otimes T$, we can give a monoid structure on $S \otimes T$.

Unit:



Multiplication:



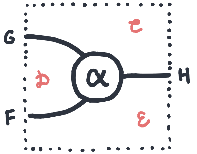
Bicategories

String Diagrams for Category Theory

As in the 1-dimensional case, we can take the POINCARÈ DUAL of the diagrams containing natural transformations.

EXAMPLE. Let $\alpha : G \circ F \Rightarrow H$ be a natural tr.

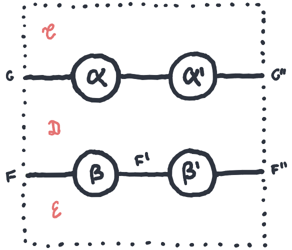
$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[rr, bend right=40, "H"', ""{name=H}] & \mathcal{D} \arrow[r, "G"] \arrow[d, phantom, "\Downarrow \alpha"] & \mathcal{E} \\
 & {} &
\end{tikzcd}$$



Equations such as the interchange law for natural transformations become invisible.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "F'" description] \arrow[r, bend right=60, "F''"'] \arrow[r, phantom, shift left=3, "\Downarrow \alpha"] \arrow[r, phantom, shift right=3, "\Downarrow \alpha'"] & \mathcal{D} \arrow[r, bend left=60, "G"] \arrow[r, "G'" description] \arrow[r, bend right=60, "G''"'] \arrow[r, phantom, shift left=3, "\Downarrow \beta"] \arrow[r, phantom, shift right=3, "\Downarrow \beta'"] & \mathcal{E}
\end{tikzcd}$$

$(\alpha ; \alpha') * (\beta ; \beta') = (\alpha * \beta) ; (\alpha' * \beta')$



REMARK. The structure where we can interpret diagrams with colored regions is a bicategory. Bicategories have 0-cells (regions), 1-cells (wires) and 2-cells (nodes); and monoidal categories are bicategories with a single 0-cell.

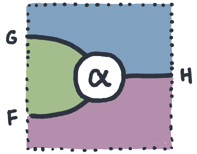
We can think of bicategories as monoidal categories with coloured regions.

String Diagrams for Category Theory

As in the 1-dimensional case, we can take the POINCARÉ DUAL of the diagrams containing natural transformations.

EXAMPLE. Let $\alpha : G \circ F \Rightarrow H$ be a natural tr.

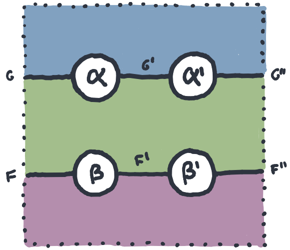
$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[rr, bend right=50, "H"', ""{name=H}] & \mathcal{D} \arrow[r, "G"] \arrow[to=H, Rightarrow, "\alpha"] & \mathcal{E}
\end{tikzcd}$$



Equations such as the interchange law for natural transformations become invisible.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F", ""{name=F, below}] \arrow[r, "F'" description, ""{name=F1}] \arrow[r, bend right=60, "F''"', ""{name=F2, above}] \arrow[from=F, to=F1, Rightarrow, "\alpha"] \arrow[from=F1, to=F2, Rightarrow, "\alpha'"] & \mathcal{D} \arrow[r, bend left=60, "G", ""{name=G, below}] \arrow[r, "G'" description, ""{name=G1}] \arrow[r, bend right=60, "G''"', ""{name=G2, above}] \arrow[from=G, to=G1, Rightarrow, "\beta"] \arrow[from=G1, to=G2, Rightarrow, "\beta'"] & \mathcal{E}
\end{tikzcd}$$

$$(\alpha ; \alpha') * (\beta ; \beta') = (\alpha * \beta) ; (\alpha' * \beta')$$

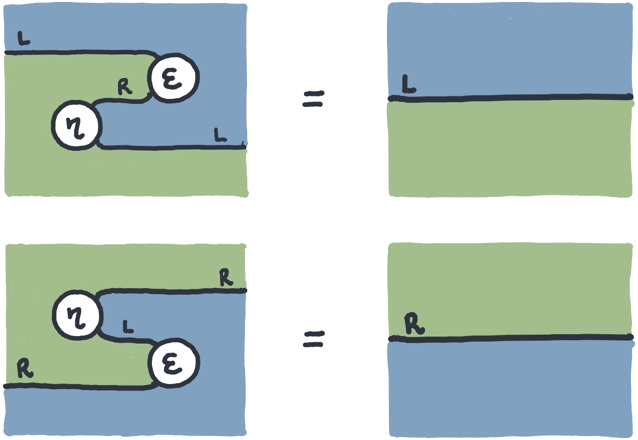


REMARK. The structure where we can interpret diagrams with colored regions is a bicategory. Bicategories have 0-cells (regions), 1-cells (wires) and 2-cells (nodes); and monoidal categories are bicategories with a single 0-cell.

We can think of bicategories as monoidal categories with coloured regions.

String Diagrams for Adjunctions

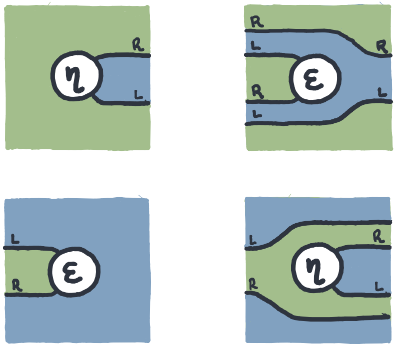
DEFINITION. In a bicategory, $L : \mathcal{X} \to \mathcal{Y}$ is left-dual to $R : \mathcal{Y} \to \mathcal{X}$, and R is right-dual to L; written $L \dashv R$; when there exist 1-cells $\varepsilon : L \otimes R \to I$ and $\eta : I \to R \otimes L$ such that



These are called the "snake equations".

THEOREM. A duality $L \dashv R$ induces a monoid structure in $R \otimes L$ and a comonoid structure in $L \otimes R$.

Proof. We consider the following unit, multiplication, counit and comultiplication.



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